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CONTRAVARIANT FUNCTORS ON THE CATEGORY OF FINITELY PRESENTED MODULES

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ABSTRACT

If R is an associative ring with identity, a theory of minimal flat resolutions is developed in the category $((R\text{-mod})^{\text{op}}, \text{Ab})$ of contravariant functors $G : (R\text{-mod})^{\text{op}} \to \text{Ab}$ from the category R-mod of finitely presented left R-modules to the category Ab of abelian groups. For a left R-module M, it is shown that the flat contravariant functor $(-, M)$ is cotorsion if and only if M is pure-injective. This is applied to characterize when a flat resolution of an object F in $((R\text{-mod})^{\text{op}}, \text{Ab})$ is minimal, and is used to construct a minimal flat resolution of F, given a projective presentation.

It is shown that the injective objects of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ are precisely those of the form $\text{Ext}^1(-, M)$, where M is pure-injective, and if $m : M \to \text{PE}(M)$ is the pure-injective envelope of M, then $\text{Ext}^1(-, m) : \text{Ext}^1(-, M) \to \text{Ext}^1(-, \text{PE}(M))$ is an injective envelope of $\text{Ext}^1(-, M)$ in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. $M \mapsto \text{Ext}^1(-, M)$ yields an explicit equivalence between the subcategory of injective objects of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ and the category of pure-injective left R-modules, modulo morphisms that factor through an injective. The characterization of minimal flat resolutions is also used to describe the relationship between the minimal flat resolution in $((R\text{-mod})^{\text{op}}, \text{Ab})$ of a functor F on the stable category and its minimal injective copresentation in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$.

A final application is a description of the contravariant Gabriel spectrum of R, the set of indecomposable injective objects of the functor category $((R\text{-mod})^{\text{op}}, \text{Ab})$. The points are in bijective correspondence with the set of pure-injective indecomposable left R-modules, which correspond to the points of the covariant Gabriel spectrum of R. It is proved that both Gabriel spectra of R may be partitioned into an open and a closed set such that this canonical bijection restricts to a homeomorphism on each.

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Let R be an associative ring with identity and denote by R-mod the category of finitely presented left R-modules. Given a left R-module M, there is an associated contravariant functor $(-, M) := \mathrm{Hom}_R(-, M) : (R\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$ with values in the category Ab of abelian groups. This rule $M \mapsto (-, M)$ constitutes a full and faithful functor from the category R-Mod of all left R-modules into the Grothendieck category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of contravariant functors $F : (R\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$. M. Auslander proposed to study the representation theory of R in terms of the ambient category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. In his work on coherent functors [1], he showed that if the category R-mod is abelian, that is, if the ring R is left coherent, then the subcategory $\mathrm{fp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of finitely presented contravariant functors is itself an abelian category.

The category R-Mod may also be embedded into the Grothendieck category $(\text{mod-}R, \mathrm{Ab})$ of covariant functors $G : \text{mod-}R \to \mathrm{Ab}$ on the category mod-R of finitely presented right R-modules. This is accomplished by associating to the left R-module M the functor $- \otimes_R M$. The category of covariant functors has several advantages over the category of contravariant functors. Auslander [2, §III.2] showed that the subcategory $\mathrm{fp}(\text{mod-}R, \mathrm{Ab})$ is abelian without any hypotheses on the ring R; Gruson and Jensen [17] characterized the injective objects of $(\text{mod-}R, \mathrm{Ab})$ as the functors isomorphic to some $-\otimes_R M$, where M is a pure-injective R-module; and Auslander [4] and Gruson and Jensen [17] noticed independently that there exists a duality between the respective subcategories $\mathrm{fp}(\text{mod-}R, \mathrm{Ab})$ and $\mathrm{fp}(R\text{-mod}, \mathrm{Ab})$ of finitely presented functors.

Perhaps the most profound application of the functorial perspective was obtained in the representation theory of an artin algebra $R = \Lambda$. Auslander and Reiten [6] analyzed the finer aspects of the category $\mathrm{fp}((\Lambda\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, and developed a theory of almost split sequences [7]. In this setting, the category of contravariant functors enjoys all the advantages of the category of covariant functors, because the duality $D : (\Lambda\text{-mod})^{\mathrm{op}} \to \text{mod-}\Lambda$ (cf. [2, §III.1]) induces an equivalence $D_* : (\text{mod-}\Lambda, \mathrm{Ab}) \to ((\Lambda\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of categories. The objective of this article is to develop a theory for the category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of contravariant functors for a general ring R that is analogous to the theory developed by Auslander and Reiten [6] for the category $\mathrm{fp}((\Lambda\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of finitely presented contravariant functors when the ring is an artin algebra Λ.

For example, a fundamental property of a finitely presented contravariant functor $F : (\Lambda\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$ is that it possesses a minimal projective resolution, which is of the form

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{(-,g)} (-,K) \xrightarrow{\pi} F \longrightarrow 0,$$

where M, N and K are finitely presented left Λ-modules. This may be generalized to the setting where R is a Krull-Schmidt ring [19], but does not hold in general. The analogous property of a general contravariant functor G in $((R\text{-mod})^{\text{op}}, \text{Ab})$ is that it possesses a minimal flat resolution. The flat objects of $((R\text{-mod})^{\text{op}}, \text{Ab})$ have been characterized by Crawley-Boevey [14, Theorem 1.4] as those functors isomorphic to $(-,M)$ for some left R-module M. The existence of flat covers in $((R\text{-mod})^{\text{op}}, \text{Ab})$ follows from a result of Saorín and Del Valle [31, Proposition 2.4]. In the first section (Theorem 10) of this article, we characterize minimal flat resolutions in $((R\text{-mod})^{\text{op}}, \text{Ab})$ as the flat resolutions having the form

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{(-,g)} (-,K) \xrightarrow{\pi} G \longrightarrow 0,$$

where M and N are pure-injective left R-modules (possibly 0) and $f : M \to N$ and $g : N \to K$ are nowhere pure morphisms (see §1.4 for the definition). We also show how pure-injective envelopes may be used to construct a minimal flat resolution of G starting with a projective presentation. The characterization of minimal flat resolutions is an easy consequence of the following theorem, which is a special case of an observation made in [31, Proposition 2.8]. Recall that a functor $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ is **cotorsion** if $\text{Ext}^1[(-,N), F] = 0$ for every flat object $(-,N)$ of $((R\text{-mod})^{\text{op}}, \text{Ab})$.

Theorem 4: *A flat functor $(-,M) \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ is cotorsion if and only if M is a pure-injective module.*

The (projectively) stable category R-$\underline{\text{mod}}$ is the category whose objects are those of R-mod, and whose morphisms are the morphisms of R-mod modulo morphisms that factor through a projective module. As in the case of R-mod, the preadditive category R-$\underline{\text{mod}}$ may be fully and faithfully embedded, via the functor $\underline{M} \mapsto \underline{\text{Hom}}(-,\underline{M})$, into the Grothendieck category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ of contravariant additive functors $G : (R\text{-}\underline{\text{mod}})^{\text{op}} \to \text{Ab}$. Auslander and Reiten [6] proved that if $R = \Lambda$ is an artin algebra and $M \in \Lambda$-mod, then the functor $\text{Ext}^1(-,M)$ is a finitely presented object of $((\Lambda\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$; they characterized these functors as the injective objects of $\mathrm{fp}((\Lambda\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. The first application of our theory of minimal flat resolutions is the following analogue in the functor category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

THEOREM 23: *The functor* $\overline{M} \mapsto \mathrm{Ext}^1(-, M)$ *is an equivalence between the category* $R\text{-}\overline{\mathrm{Pinj}}$ *of pure-injective left* R*-modules, modulo morphisms that factor through an injective module, and the subcategory of injective objects of* $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

That the two categories mentioned in Theorem 23 are equivalent is a special case of a result of H. Krause [24, Theorem 5.3]; Theorem 23 provides an explicit equivalence.

Gruson and Jensen [17] proved that if M is an R-module and $m\colon M \to \mathrm{PE}(M)$ is the pure-injective envelope of M, then the induced morphism

$$- \otimes m : - \otimes_R M \to - \otimes_R \mathrm{PE}(M)$$

is the injective envelope in the category $(\mathrm{mod}\text{-}R, \mathrm{Ab})$ of the object $- \otimes_R M$. Theorem 31 shows that the induced morphism

$$\mathrm{Ext}^1(-, m) : \mathrm{Ext}^1(-, M) \to \mathrm{Ext}^1(-, \mathrm{PE}(M))$$

is the injective envelope of $\mathrm{Ext}^1(-, M)$ in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

For an artin algebra Λ, Auslander and Reiten [6, Proposition 4.6] showed that if an object G in $\mathrm{fp}((\Lambda\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ has the minimal projective resolution

$$0 \longrightarrow (-, M) \xrightarrow{(-,f)} (-, N) \xrightarrow{(-,g)} (-, K) \xrightarrow{\pi} G \longrightarrow 0$$

in $\mathrm{fp}((\Lambda\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, then the minimal injective copresentation of G in $\mathrm{fp}((\Lambda\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is given by

$$0 \longrightarrow G \longrightarrow \mathrm{Ext}^1(-, M) \xrightarrow{\mathrm{Ext}^1(-,f)} \mathrm{Ext}^1(-, N).$$

This minimal injective copresentation is obtained from the next two terms of the long exact sequence of functors associated to the extension in $\mathrm{Ext}^1(K, M)$ that induces the minimal projective resolution of G. Another application of our theory of minimal flat resolutions, Corollary 33, asserts the same property for an object of the category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ for a general ring R, with the minimal projective resolution replaced by a minimal flat resolution. This result has a converse (Corollary 34), which shows how a minimal injective copresentation in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ of an object G may be used to obtain the minimal flat resolution of G in $((R\text{-mod})^{\text{op}}, \text{Ab})$.

The projection functor $\pi : R\text{-mod} \to R\text{-}\underline{\text{mod}}$ induces a full and faithful functor

$$((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab}) \subseteq ((R\text{-mod})^{\text{op}}, \text{Ab})$$

that permits one to identify the category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ with the subcategory of $((R\text{-mod})^{\text{op}}, \text{Ab})$ of functors G that vanish on the regular representation ${}_RR$, $G(R) = 0$. The subcategory $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is a torsion class of $((R\text{-mod})^{\text{op}}, \text{Ab})$. Similarly, the category $(\underline{\text{mod}}\text{-}R, \text{Ab})$ may be viewed as the torsion class of $(\text{mod-}R, \text{Ab})$ consisting of functors that vanish on R_R. The injective objects of $(\underline{\text{mod}}\text{-}R, \text{Ab})$ have the form $t(- \otimes_R M)$, where M is a pure-injective R-module and $t(-)$ is the torsion subfunctor. We use the dual of an argument found in [7] to prove the following.

Theorem 29: *The Auslander-Bridger Transpose* $\text{Tr} : \underline{\text{mod}}\text{-}R \to (R\text{-}\underline{\text{mod}})^{\text{op}}$ *induces an equivalence of categories* $\text{Tr}_* : ((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab}) \to (\underline{\text{mod}}\text{-}R, \text{Ab})$ *with the property that*

$$\text{Tr}_*[\text{Ext}^1(-, M)] \cong t(- \otimes M),$$

for every R-module M.

The Gabriel spectrum of $((R\text{-mod})^{\text{op}}, \text{Ab})$ is the space $\text{Sp}((R\text{-mod})^{\text{op}}, \text{Ab})$ whose points are the indecomposable injective objects of $((R\text{-mod})^{\text{op}}, \text{Ab})$, up to isomorphism. Theorem 23 may be used to give a complete list, without repetition, of the points of the Gabriel spectrum $\text{Sp}((R\text{-mod})^{\text{op}}, \text{Ab})$. They fall into 2 classes:

- **With torsion:** $E[\text{Ext}^1(-, M)]$, the injective envelope of $\text{Ext}^1(-, M)$, where M is an indecomposable pure-injective left R-module that is not injective;
- **Torsion-free:** $\text{Hom}(-, E)$, where E is an indecomposable injective left R-module.

This classification of the indecomposable injective objects of $((R\text{-mod})^{\text{op}}, \text{Ab})$ yields a canonical bijective correspondence Ξ_R between $\text{Sp}(\text{mod-}R, \text{Ab})$ and $\text{Sp}((R\text{-mod})^{\text{op}}, \text{Ab})$ given by

$$\Xi_R : - \otimes_R U \mapsto \begin{cases} E[\text{Ext}^1(-, U)], & \text{if } U \text{ is not injective;} \\ (-, U), & \text{if } U \text{ is injective.} \end{cases} \tag{1}$$

Since the subcategory $(\underline{\text{mod}}\text{-}R, \text{Ab}) \subseteq (\text{mod-}R, \text{Ab})$ is a hereditary torsion class, it induces a partition of the Gabriel spectrum

$$\text{Sp}(\text{mod-}R, \text{Ab}) = \text{Sp}(\underline{\text{mod}}\text{-}R, \text{Ab}) \mathbin{\dot\cup} \text{Sp}(R\text{-Mod})$$

into an open and a closed subsets. The hereditary torsion class $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ induces a similar partition of $\text{Sp}((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. The canonical bijective correspondence Ξ_R respects these partitions (Theorem 36) and restricts to a homeomorphism between the open subsets and closed subsets, respectively. If the ring R is left coherent, then the categories $((R\text{-mod})^{\text{op}}, \text{Ab})$ and $(\text{mod-}R, \text{Ab})$ are locally coherent Grothendieck categories, and the correspondence Ξ_R induces a similar relationship between the covariant and contravariant Ziegler spectra of R. If R is left semihereditary, then Proposition 51 shows that Ξ_R is a homeomorphism between $\text{Sp}(\text{mod-}R, \text{Ab})$ and $\text{Sp}((R\text{-mod})^{\text{op}}, \text{Ab})$.

Let $R\text{-}\overline{\text{Mod}}$ be the category whose objects are the left R-modules, and whose morphisms are the morphisms of R-Mod modulo those that factor through an fp-injective module. If R is a Quasi-Frobenius ring, then the category $R\text{-}\overline{\text{Mod}}$ is the category of stable modules and may be equipped with the structure of a triangulated category [11, 35, 36]. A theory of purity has been developed for triangulated categories by Beligiannis [8, 9] and H. Krause [25]. In Section 4, Theorem 23 is used to introduce notions in $R\text{-}\overline{\text{Mod}}$ that generalize those of pure-monomorphism and pure-injective object in the Quasi-Frobenius case. It is shown that every object $\overline{M}$ of $R\text{-}\overline{\text{Mod}}$ admits an $R\text{-}\overline{\text{Pinj}}$-envelope, where $R\text{-}\overline{\text{Pinj}} \subseteq R\text{-}\overline{\text{Mod}}$ is the subcategory of classes with a pure-injective representative. Considered as a morphism, the $R\text{-}\overline{\text{Pinj}}$ envelope $\overline{m} : \overline{M} \to \text{PE}(\overline{M})$ is an Ext-monomorphism, that is, $\text{Ext}^1(-, m)$ is a monomorphism in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. If R is Quasi-Frobenius, the objects of $R\text{-}\overline{\text{Pinj}}$ are the pure-injective objects of $R\text{-}\overline{\text{Mod}}$, considered as a triangulated category, and the Ext-monomorphisms are the pure-monomorphisms of $R\text{-}\overline{\text{Mod}}$.

If R-Mod is embedded into the category $((R\text{-mod})^{\text{op}}, \text{Ab})$ via the rule $M \mapsto (-, M)$, then its objects can be characterized as the flat objects of $((R\text{-mod})^{\text{op}}, \text{Ab})$. Similarly, the objects of $(\text{mod-}R, \text{Ab})$ that are isomorphic to $- \otimes_R M$, for some R-module M, may be homologically characterized as the fp-injective objects of $(\text{mod-}R, \text{Ab})$. In Section 5, we show (Corollary 44) that if the ring R is left coherent, then the objects $\text{Ext}^1(-, M)$ are fp-injective in the category $((R\text{-mod})^{\text{op}}, \text{Ab})$. This leads to a characterization of the left coherent rings R for which the category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is locally noetherian.

THEOREM 48: *The following conditions are equivalent for a left coherent ring R:*

(1) *for every module M, the quotient* $\text{PE}(M)/M$ *is fp-injective;*

(2) *for every module M, the object* $\text{Ext}^1(-, M)$ *is injective in*

$$((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab});$$

(3) *the category* $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ *is locally noetherian.*

Section 5 also contains a brief treatment of Garavaglia's notion [30] of elementary Krull dimension. It is shown (Theorem 47) that if a left R-module M has finite elementary Krull dimension n, then its pure-injective dimension is bounded by n.

Throughout the article, R denotes an associative ring with identity and $J(R)$ the Jacobson radical of R. The unadorned term ***R*-module** will refer to a unital **left** R-module. The category of R-modules is denoted by R-Mod; the category of abelian groups by Ab. The category of finitely presented (resp., right) R-modules is denoted by R-mod (resp., mod-R). Given R-modules M and N, we use the abbreviation $(M, N) := \text{Hom}_R(M, N)$; the unqualified tensor product $\otimes$ always refers to the tensor $\otimes_R$ over R. The category of contravariant additive functors $F : (R\text{-mod})^{\text{op}} \to \text{Ab}$ is denoted by $((R\text{-mod})^{\text{op}}, \text{Ab})$; the category of covariant functors by $(\text{mod-}R, \text{Ab})$. If F and G are objects of $((R\text{-mod})^{\text{op}}, \text{Ab})$, then the set of morphisms $[F, G]$ consists of the natural transformations from F to G. The notation $F : \mathcal{A} \to \mathcal{B}$ for a functor between categories $\mathcal{A}$ and $\mathcal{B}$ is reserved for **covariant** functors. Thus a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is indicated by $G : \mathcal{A}^{\text{op}} \to \mathcal{B}$.

1. Minimal flat resolutions

In this section, we will characterize minimal flat resolutions in $((R\text{-mod})^{\text{op}}, \text{Ab})$ and show how a projective presentation of an object $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ may be used to construct a minimal flat resolution of F.

1.1. FLAT FUNCTORS. A functor $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ is called **representable** if it is isomorphic to a functor of the form $(-, A)$, where A is a finitely presented module.

Yoneda's Lemma: (cf. [33, Proposition IV.7.3]) *Let A be a finitely presented R-module and $F \in ((R\text{-mod})^{\rm op}, \mathrm{Ab})$. There is an isomorphism*

$$\Theta_{(A,F)} : [(-, A), F] \to F(A)$$

natural in both F and A.

A sequence of functors $F \xrightarrow{\mu} G \xrightarrow{\nu} H$ is exact in $((R\text{-mod})^{\rm op}, \mathrm{Ab})$ if for every finitely presented R-module A, the corresponding sequence of abelian groups

$$F(A) \xrightarrow{\mu(A)} G(A) \xrightarrow{\nu(A)} H(A)$$

is exact. By Yoneda's Lemma, every representable object of $((R\text{-mod})^{\rm op}, \mathrm{Ab})$ is projective. Indeed, since the representable functors form a generating set [33, Corollary 7.5], an object $G \in ((R\text{-mod})^{\rm op}, \mathrm{Ab})$ is **projective** if and only if it is isomorphic to a coproduct factor of a coproduct of representable functors. A finite coproduct of representable functors is representable and since idempotents split in R-mod, a functor in $((R\text{-mod})^{\rm op}, \mathrm{Ab})$ is representable if and only if it is a finitely generated projective object of $((R\text{-mod})^{\rm op}, \mathrm{Ab})$.

Proposition 1 ([33, Corollary 7.4]): *The Yoneda functor*

$$\Upsilon : R\text{-mod} \to ((R\text{-mod})^{\rm op}, \mathrm{Ab}),$$

given by $A \mapsto (-, A)$, is a full and faithful left exact functor. It is an equivalence between the category R-mod of finitely presented R-modules and the category $\mathrm{proj}((R\text{-mod})^{\rm op}, \mathrm{Ab})$ of finitely generated projective objects of $((R\text{-mod})^{\rm op}, \mathrm{Ab})$.

As in the case of exactness, a direct limit $\varinjlim F_i$ of functors in $((R\text{-mod})^{\rm op}, \mathrm{Ab})$ is calculated objectwise. Thus if $A \in R$-mod, then

$$(\varinjlim F_i)(A) := \varinjlim F_i(A),$$

where the direct limit on the right is taken in the category Ab of abelian groups. An object $G \in ((R\text{-mod})^{\rm op}, \mathrm{Ab})$ is **flat** if it is isomorphic to a direct limit of finitely generated projective functors,

$$G \cong \varinjlim (-, A_i).$$

The (full) subcategory of $((R\text{-mod})^{\rm op}, \mathrm{Ab})$ of flat functors is denoted by $\mathrm{Flat}((R\text{-mod})^{\rm op}, \mathrm{Ab})$. Because the Yoneda functor is an equivalence between R-mod and the category of representable functors, we may consider the corresponding direct limit $M = \varinjlim A_i$ in R-Mod. A well-known characterization [33, Proposition V.3.4] of finitely presented modules implies that

$$G \cong \varinjlim(-, A_i) \cong (-, \varinjlim A_i) \cong (-, M).$$

PROPOSITION 2 ([14, Theorem 1.4]): *The functor* $\Upsilon : R\text{-Mod} \to ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, *given by* $M \mapsto (-, M)$, *is a full and faithful left exact functor. It yields an equivalence between the category* R-Mod *of* R*-modules and the category* $\mathrm{Flat}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.

Proposition 2 may be used to see that the category $\mathrm{Flat}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is closed under direct limits, direct products and coproduct factors in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. Every projective object of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is flat.

In general, the category R-mod is not abelian. However, it has cokernels, so we may call a contravariant functor $F : (R\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$ **left exact** if it takes cokernels to kernels. Since every flat functor in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is isomorphic to a functor of the form $(-, M)$, it is left exact. Conversely, let us note that, if $F : (R\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$ is left exact, then $F \cong (-, F(R))$ is flat. First, there is an obvious isomorphism $\alpha(R) : F(R) \to (R, F(R))$ of abelian groups; it induces for every finitely generated free module R^n an isomorphism $\alpha(R^n) : F(R^n) \to (R, F(R^n))$. Given $A \in R$-mod, consider a free presentation

$$R^m \longrightarrow R^n \longrightarrow A \longrightarrow 0$$

and apply the two functors F and $(-, F(R))$. Then, an isomorphism $\alpha(A) : F(A) \to (A, F(R))$ is induced by left exactness,

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(A) & \longrightarrow & F(R^n) & \longrightarrow & F(R^m) \\ & & \downarrow{\scriptstyle \alpha(A)} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & (A, F(R)) & \longrightarrow & (R^n, F(F)) & \longrightarrow & (R^m, F(R)). \end{array}$$

This observation together with a routine diagram chase may be used to show the following.

PROPOSITION 3: *If the sequence in* $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$

$$0 \longrightarrow (-, M) \longrightarrow F \longrightarrow (-, K) \longrightarrow 0$$

is exact, then F *is flat.*

Let n be a whole number. An object $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ is said to be of **flat dimension at most n** if there is a flat resolution

$$\cdots \longrightarrow (-, M_2) \longrightarrow (-, M_1) \longrightarrow (-, M_0) \xrightarrow{D_0} F \longrightarrow 0$$

with $M_{n+1} = 0$. Consider a projective presentation

$$(-, M_1) \xrightarrow{\eta} (-, M_0) \xrightarrow{\pi} F \longrightarrow 0$$

in $((R\text{-mod})^{\text{op}}, \text{Ab})$ of the functor F. By Proposition 2, there is an R-morphism $d_1 : M_1 \to M_0$ such that $\eta = (-, d_1)$. Let $d_2 : M_2 \to M_1$ be the kernel of d_1. Because the functor Υ is left exact, the sequence

$$0 \longrightarrow (-, M_2) \xrightarrow{(-, d_2)} (-, M_1) \xrightarrow{(-, d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0$$

yields an exact sequence in $((R\text{-mod})^{\text{op}}, \text{Ab})$, which is a flat resolution of F of length at most 2. Thus every object in the category $((R\text{-mod})^{\text{op}}, \text{Ab})$ has flat dimension at most 2.

Let $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$. A morphism $\varphi : (-, M) \to F$ from a flat functor is a **flat precover** if, given a morphism $\psi : (-, N) \to F$ whose domain is a flat object $(-, N)$, there is an R-morphism $f : N \to M$ such that the diagram

$$\begin{array}{ccc} & & (-, N) \\ & \overset{(-,f)}{\swarrow} & \downarrow \psi \\ (-, M) & \xrightarrow{\varphi} & F \end{array}$$

commutes. A flat precover $\varphi : (-, M) \to F$ is a **flat cover** if any R-endomorphism $g : M \to M$ for which the diagram

$$\begin{array}{ccc} & & (-, M) \\ & \overset{(-,g)}{\swarrow} & \downarrow \varphi \\ (-, M) & \xrightarrow{\varphi} & F \end{array}$$

commutes is an automorphism of M.

1.2. Cotorsion functors. A functor $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ is **cotorsion** if $\text{Ext}^1[(-, M), F] = 0$ for every flat object $(-, M)$ of $((R\text{-mod})^{\text{op}}, \text{Ab})$. Any undefended claim that we make about cotorsion objects in $((R\text{-mod})^{\text{op}}, \text{Ab})$

has a proof similar to the corresponding fact about cotorsion modules, and may be found in [37]. If F is cotorsion, then $\mathrm{Ext}^i[(-,M),F] = 0$ for all $i \geq 1$. This may be verified with the help of the argument above, which shows that in a projective resolution of a flat functor $(-,M)$, all the syzygies are flat. Consequently, if F is a cotorsion object and

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

is a short exact sequence in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, then G is cotorsion if and only if H is cotorsion.

If F is cotorsion in the short exact sequence

$$0 \longrightarrow F \longrightarrow (-,M_0) \xrightarrow{D_0} G \longrightarrow 0,$$

then $D_0 : (-,M_0) \to G$ is a flat precover of G. Dually, if $G \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is arbitrary and there is a cotorsion object F such that the short exact sequence

$$0 \longrightarrow G \xrightarrow{\eta} F \longrightarrow (-,M) \longrightarrow 0$$

is exact, then the morphism $\eta : G \to F$ is a cotorsion preenvelope; a **preenvelope** is defined in a manner dual to that of a precover [37, §1.2].

In order to characterize flat cotorsion functors, let us recall the notion of a pure-injective module. A short exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0$$

is **pure-exact** if the corresponding sequence in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{(-,g)} (-,K) \longrightarrow 0$$

is exact. In that case, the morphism $f : M \to N$ is called a **pure-monomorphism.** A submodule $M \subseteq N$ is called a **pure** submodule if the inclusion morphism $\iota : M \to N$ is a pure-monomorphism. The morphism g in a pure-exact sequence is called a **pure-epimorphism.** Thus a morphism g is a pure-epimorphism if and only if for every finitely presented module A, the morphism $(A,g) : (A,N) \to (A,K)$ of abelian groups is an epimorphism.

A module M is **pure-injective** if every pure-monomorphism $f : M \to N$ is a split monomorphism. Every module M admits a **pure-injective envelope** $m : M \to \mathrm{PE}(M)$ [23], which is unique up to isomorphism over M. Dually, a module K is called **pure-projective** if every pure-epimorphism is a split epimorphism. It is clear from the definition that every finitely presented module is pure-projective.

On several occasions, we will need the fact that the endomorphism ring $S = \mathrm{End}_R M$ of a pure-injective module M is an **exchange ring** [27, §1.3]. This follows from the characterization [28, Theorem 2.1] of exchange rings due to Nicholson and is verified in [18, Lemma 2]. It implies [28, Proposotion 1.9] that an endomorphism $f : M \to M$ belongs to the Jacobson radical $J(S)$ if and only if the principal left ideal Sf contains no nonzero idempotent elements.

THEOREM 4 (cf. [31, Proposition 2.8]): *A flat functor $(-, M) \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is cotorsion if and only if M is a pure-injective module. If M is an R-module and $m : M \to \mathrm{PE}(M)$ is the pure-injective envelope, then $(-, m) : (-, M) \to (-, \mathrm{PE}(M))$ is the cotorsion envelope of the flat functor $(-, M)$.*

Proof. Suppose that $(-, M)$ is a cotorsion object, and consider the pure-injective envelope $m : M \to \mathrm{PE}(M)$. The short exact sequence

$$0 \longrightarrow M \xrightarrow{m} \mathrm{PE}(M) \longrightarrow \mathrm{PE}(M)/M \longrightarrow 0$$

is pure-exact, so the corresponding sequence in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$,

$$0 \longrightarrow (-, M) \longrightarrow (-, \mathrm{PE}(M)) \longrightarrow (-, \mathrm{PE}(M)/M) \longrightarrow 0$$

is exact. As $(-, \mathrm{PE}(M)/M)$ is flat, the sequence splits and $M = \mathrm{PE}(M)$ is pure-injective.

For the converse, suppose that M is a pure-injective module and consider an extension

$$0 \longrightarrow (-, M) \xrightarrow{\mu} G \xrightarrow{\nu} (-, Z) \longrightarrow 0$$

of $(-, M)$ by a flat functor $(-, Z)$. By Proposition 3, G is isomorphic to the flat functor $(-, G(R))$. Replacing G by $(-, G(R))$, we get that $\mu = (-, f)$ and $\nu = (-, g)$, where

$$0 \longrightarrow M \xrightarrow{f} G(R) \xrightarrow{g} Z \longrightarrow 0$$

is a pure-exact sequence. As M is pure-injective, the sequence splits.

To prove the second statement, let M be a left R-module. The morphism of functors $(-,m) : (-,M) \to (-,\mathrm{PE}(M))$ is a monomorphism into a cotorsion functor $(-,\mathrm{PE}(M))$ whose cokernel is the flat object $(-,\mathrm{PE}(M)/M)$. Thus the morphism $(-,m) : (-,M) \to (-,\mathrm{PE}(M))$ is a cotorsion preenvelope. To see that it is a cotorsion envelope, just note that any endomorphism of $(-,\mathrm{PE}(M))$ over $(-,M)$ is of the form $(-,g)$, where $g : \mathrm{PE}(M) \to \mathrm{PE}(M)$ is an endomorphism over M. As $\mathrm{PE}(M)$ is the pure-injective envelope of M, the endomorphism g must be an automorphism. ∎

1.3. Special resolutions. A flat precover is called **special** [15, p. 153] if it has a cotorsion kernel. Given an object F in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, a flat resolution

$$0 \longrightarrow (-,M_2) \xrightarrow{(-,d_2)} (-,M_1) \xrightarrow{(-,d_1)} (-,M_0) \xrightarrow{D_0} F \longrightarrow 0$$

of F will be called **special** if both morphisms $D_0 : (-,M_0) \to F$ and $(-,d_1) : (-,M_1) \to \mathrm{Im}(-,d_1)$ are special flat precovers.

Proposition 5: *The flat resolution of F given above is special if and only if M_1 and M_2 are pure-injective.*

Proof. The kernel of $(-,d_1)$ is cotorsion if and only if M_2 is pure-injective. Assuming that M_2 is pure-injective, then the $\mathrm{Ker}\, D_0 = \mathrm{Im}(-,d_1)$ is cotorsion if and only if $(-,M_1)$ is cotorsion if and only if M_1 is pure-injective. ∎

Suppose that G has flat dimension at most 1. There is a flat resolution of the form

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{\pi} G \longrightarrow 0.$$

Evaluating the resolution at ${}_RR$ shows that $f : M \to N$ is a monomorphism. Let $g : N \to K$ be the cokernel of f; one obtains an exact sequence of functors

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{(-,g)} (-,K).$$

Because G is the cokernel of $(-,f)$, it may be embedded into the flat functor $(-,K)$.

Conversely, suppose that there is a monomorphism $\iota : G \subseteq (-,K)$ of G into a flat functor, and consider an epimorphism $\eta : (-,N) \to G$ from a projective functor. The composition is of the form $(-,g) : (-,N) \to (-,K)$. If $f : M \to N$

is the kernel of g, one obtains a flat resolution of G as above. Thus the flat dimension of G is at most 1. We have proved the following.

PROPOSITION 6: *An object* $G \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ *is of flat dimension at most* 1 *if and only if there is a monomorphism* $\iota : G \to (-, K)$ *into a flat object.*

The flat resolution of G above may be used to produce a special flat resolution, if we take the pushout with the cotorsion envelope $(-, m)\colon (-, M) \to (-, \mathrm{PE}(M))$. One obtains the following commutative diagram with exact rows and columns,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (-, M) & \xrightarrow{(-,f)} & (-, N) & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & (-, \mathrm{PE}(M)) & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & (-, Z) & = & (-, Z) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

where $Z = \mathrm{PE}(M)/M$. By Proposition 3, the functor $F \cong (-, N')$ is flat. Because $(-, \mathrm{PE}(M))$ is cotorsion, the flat resolution of G given in the middle row is special.

A special resolution of length 1 may be used to produce a cotorsion preenvelope of G. Let $n : N \to \mathrm{PE}(N)$ be the pure-injective envelope and consider the commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (-, M) & \xrightarrow{(-,f)} & (-, N) & \xrightarrow{\pi} & G & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (-,n)} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & (-, M) & \xrightarrow{(-,nf)} & (-, \mathrm{PE}(N)) & \longrightarrow & G' & \longrightarrow & 0.
\end{array}$$

Both M and $\mathrm{PE}(N)$ are pure-injective, so that G' is cotorsion. The morphism $\eta : G \to G'$ is clearly a monomorphism and because the right square is a pullback-pushout diagram, $\operatorname{Coker}\eta = \operatorname{Coker}(-,n) \cong (-,\mathrm{PE}(N)/N)$ is flat. Thus the morphism $\eta : G \to G'$ is a cotorsion preenvelope. The next proposition may also be deduced from [31, Proposition 2.4].

Proposition 7: *Every object $F \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ has a special flat resolution.*

Proof. Consider an epimorphism $\eta : (-,K') \to F$ from a projective object. The kernel $G = \operatorname{Ker}\eta$ is of flat dimension at most 1. By the foregoing observations, there is a cotorsion preenvelope $\iota : G \to G'$ with a flat cokernel. The pushout construction yields a morphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\beta} & (-,K') & \xrightarrow{\eta} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & G' & \xrightarrow{\beta'} & H & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

where the morphism $\alpha : (-,K') \to H$ is a monomorphism whose cokernel is isomorphic to the flat functor $\operatorname{Coker}\iota$. By Proposition 3, $H = (-,K)$ is flat. As G' is of flat dimension at most 1, it has a special flat resolution of length at most 1 and, because G' is cotorsion, this special flat resolution consists of flat cotorsion objects. This provides a special flat resolution of F. ∎

1.4. Nowhere pure morphisms. A morphism $f : M \to N$ will be called **nowhere pure** if there is no nonzero pure submodule $M' \subseteq M$ such that the restriction of f to M' is a pure-monomorphism.

Lemma 8: *A composition of morphisms $f = gh : M \xrightarrow{h} N \xrightarrow{g} K$ is nowhere pure, if one of g and h is nowhere pure.*

Proof. Suppose that f is not nowhere pure. Then there is a pure submodule $M' \subseteq M$ such that the restriction of f to M' is a pure-monomorphism. The restriction of h to M' must also be a pure-monomorphism. It follows that the image $h(M') \subseteq N$ is a pure submodule and that the restriction of g to $h(M')$ is also a pure-monomorphism. ∎

If M is a pure injective module and $f : M \to N$ is an R-morphism, then one may apply Zorn's Lemma to obtain a pure submodule $M' \subseteq M$ maximal with respect to the property that the restriction of f to M' is a pure-monomorphism. Since the pure-injective envelope $\mathrm{PE}(M')$ is a pure-essential extension of M', the restriction of f to $\mathrm{PE}(M')$ is also a pure-monomorphism. It follows from the maximality of M' that $M' = \mathrm{PE}(M')$ is a direct summand of M,

$$M = M' \oplus M'',$$

where the restriction of f to M' is a pure-monomorphism and its restriction to M'' is nowhere pure.

PROPOSITION 9: *Let M be a pure-injective module and $S = \mathrm{End}_R M$. A morphism $f : M \to N$ is a nowhere pure morphism if and only if for every morphism $g : N \to M$, the composition $gf \in J(S)$. Equivalently, the morphism $f : M \to N$ is not nowhere pure if and only if there is a morphism $g : N \to M$ such that $gf = e$ is a nonzero idempotent element of S. In particular, an endomorphism $f : M \to M$ is nowhere pure if and only if $f \in J(S)$.*

Proof. The foregoing comments indicate that a morphism $f : M \to N$ from a pure-injective module M is a nowhere pure morphism if and only if there is no nonzero direct summand M' of M such that the restriction of f to M' is a pure-monomorphism. Equivalently, for every morphism $g : N \to M$, the left ideal Sgf contains no idempotent elements. Since the endomorphism ring $S = \mathrm{End}_R M$ is an exchange ring, this is equivalent to $gf \in J(S)$. ∎

A flat resolution of F

$$0 \longrightarrow (-, M_2) \xrightarrow{(-, d_2)} (-, M_1) \xrightarrow{(-, d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0$$

is called **minimal** if the morphisms

$$D_0 : (-, M_0) \to F \quad \text{and} \quad (-, d_1) : (-, M_1) \to \mathrm{Im}(-, d_1)$$

are flat covers. It follows from the definition of a flat cover that a minimal flat resolution $\{(-, M_i)\}_i$ of F admits a morphism from any other flat resolution of F, and that an endomorphism $\{(-, f_i)\}_i$ of $\{(-, M_i)\}_i$ which induces the identity 1_F on F is necessarily an automorphism. Thus any two minimal flat resolutions of F are isomorphic over F, and every special flat resolution of F contains a minimal one as a direct summand.

THEOREM 10: *Let $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$. A special resolution of F*

$$0 \longrightarrow (-, M_2) \xrightarrow{(-,d_2)} (-, M_1) \xrightarrow{(-,d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0$$

is minimal if and only if d_1 and d_2 are nowhere pure.

Proof. Suppose that d_2 is not nowhere pure. As M_2 is pure-injective, $d_2 : M_2 \to M_1$ is a pure-monomorphism on some nonzero direct summand of M_2. The morphism $(-, d_1)$ then contains a nonzero summand in its kernel. It cannot be the flat cover of its image, contradicting minimality. That d_1 is nowhere pure is proved similarly.

Suppose now that d_1 and d_2 are both nowhere pure. Since the given flat resolution is special, it suffices to show that any endomorphism

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & (-, M_2) & \xrightarrow{(-,d_2)} & (-, M_1) & \xrightarrow{(-,d_1)} & (-, M_0) & \xrightarrow{D_0} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (-,f_2)} & & \downarrow{\scriptstyle (-,f_1)} & & \downarrow{\scriptstyle (-,f_0)} & & \| & & \\ 0 & \longrightarrow & (-, M_2) & \xrightarrow{(-,d_2)} & (-, M_1) & \xrightarrow{(-,d_1)} & (-, M_0) & \xrightarrow{D_0} & F & \longrightarrow & 0 \end{array}$$

is an automorphism. Because the resolution is special, the endomorphism $\{(-, f_i)\}_i$ is homotopic to the identity morphism; the homotopy is given by R-morphisms $s_i : M_i \to M_{i+1}$ satisfying the equations

$$f_2 - 1_{M_2} = s_1 d_2, \quad f_1 - 1_{M_1} = d_2 s_1 + s_0 d_1.$$

The morphisms d_1 and d_2 are nowhere pure so Lemma 8 implies that $s_1 d_2$, $d_2 s_1$, and $s_0 d_1$ are also nowhere pure. From Proposition 9, we get that $f_2 - 1_{M_2} \in J(\text{End}_R M_2)$ and $f_1 - 1_{M_1} \in J(\text{End}_R M_1)$. Thus f_1 and f_2 are isomorphism, and, because $(-, f_1)$ and $(-, f_2)$ are isomorphisms, so is $(-, f_0)$. ∎

Let us use Theorem 10 and the knowledge that a minimal flat resolution must be a summand of any special flat resolution to find a minimal flat resolution. By Proposition 7, every functor $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ has a special resolution

$$0 \longrightarrow (-, M_2) \xrightarrow{(-,d_2)} (-, M_1) \xrightarrow{(-,d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0.$$

There is a direct sum decomposition $M_2 = M_2' \oplus M_2''$, where the restriction of d_2 to M_2' is a pure-monomorphism and the restriction to M_2'' is nowhere pure.

Then $d_2(M_2')$ is a direct summand of M_1, and we may consider the given special flat resolution modulo the resolution of 0 given by

$$0 \longrightarrow (-, M_2') \xrightarrow{(-,d_2)} (-, d_2(M_2')) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0.$$

The quotient is also a special resolution of F with the further property that the morphism d_2 is now nowhere pure. Similarly, there is a direct sum decomposition $M_1 = M_1' \oplus M_1''$ such that the restriction of d_1 to M_1' is a pure-monomorphism and the restriction to M_1'' is nowhere pure. Then $d_1(M_1')$ is a direct summand of M_0 and we may factor out by the resolution

$$0 \longrightarrow 0 \longrightarrow (-, M_1') \xrightarrow{(-,d_1)} (-, d_1(M_1')) \longrightarrow 0 \longrightarrow 0$$

of 0 to obtain a special resolution of F in which both d_1 and d_2 are nowhere pure. These considerations complete our proof of [31, Proposition 2.4] in the special case of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.

THEOREM 11: *Every object $F \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ has a minimal flat resolution.*

2. Functors on the stable category

An R-module Z is **flat** if every epimorphism $g : Y \to Z$ is a pure-epimorphism. As in the case of functors, flat modules may be characterized (cf. [14, Theorem 1.4]) as direct limits of finitely generated projective modules. It follows that every morphism $f : A \to Z$ from a finitely presented module A to a flat module Z factors through a finitely generated projective module.

Given R-modules M and N, let $\mathrm{Flat}(M, N) \subseteq \mathrm{Hom}_R(M, N)$ be the subgroup of morphisms that factor through a flat module. The category R-$\underline{\mathrm{Mod}}$ is the additive category whose objects are of the form $\underline{M}$, where M is an R-module, and whose morphism groups are given by

$$\underline{\mathrm{Hom}}_R(\underline{M}, \underline{N}) = \mathrm{Hom}_R(M, N)/\mathrm{Flat}(M, N).$$

The **(projectively) stable** category of finitely presented R-modules is the full subcategory R-$\underline{\mathrm{mod}} \subseteq R$-$\underline{\mathrm{Mod}}$ of classes $\underline{M}$ represented by finitely presented modules, modulo morphisms that factor through a flat module. This is equivalent to the standard definition [5], in which the morphism groups are defined to be the quotient groups $\mathrm{Hom}(A, B)/\mathrm{proj}(A, B)$, where $\mathrm{proj}(A, B)$ is the subgroup of morphisms that factor through a finitely generated projective.

2.1. The torsion-free class $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. The projection functor

$$\pi : R\text{-mod} \to R\text{-}\underline{\text{mod}}$$

is defined by $\pi(A) = \underline{A}$ on objects, and takes a morphism $f : A \to B$ to its class modulo $\text{Flat}(A, B) = \text{proj}(A, B)$. Every contravariant functor

$$G : (R\text{-}\underline{\text{mod}})^{\text{op}} \to \text{Ab}$$

induces a contravariant functor $F = G \circ \pi : (R\text{-mod})^{\text{op}} \to \text{Ab}$ with the property that $F(R) = 0$. Conversely, if $F : (R\text{-mod})^{\text{op}} \to \text{Ab}$ is a contravariant functor on R-mod such that $F(R) = 0$, then $F(P) = 0$ for every finitely generated projective P, and hence $F(f) = 0$ for every $f \in \text{proj}(A, B)$. The functor F thus induces a functor $G : (R\text{-}\underline{\text{mod}})^{\text{op}} \to \text{Ab}$ with the property that $F = G \circ \pi$. We may therefore identify the category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ of contravariant functors on $R\text{-}\underline{\text{mod}}$ with the full subcategory of $((R\text{-mod})^{\text{op}}, \text{Ab})$ whose objects are the contravariant functors that vanish on the R-module R. This subcategory $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is a Grothendieck category in its own right, and a sequence of morphisms in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is exact if and only if it is such in $((R\text{-mod})^{\text{op}}, \text{Ab})$.

Since the objects of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ are the contravariant functors on R-mod that vanish on R, they may be characterized as the objects F of $((R\text{-mod})^{\text{op}}, \text{Ab})$ with the property that for any (some) flat presentation

$$(-, M_1) \xrightarrow{(-, d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0,$$

the morphism $d_1 : M_1 \to M_0$ is an epimorphism.

Theorem 12: *If* $F \in ((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ *and* $D_0 : (-, M_0) \to F$ *is the flat cover of* F *in* $((R\text{-mod})^{\text{op}}, \text{Ab})$, *then the module* M_0 *is cotorsion.*

Proof. The flat cover is part of a minimal flat resolution of F, necessarily of length 2,

$$0 \longrightarrow (-, M_2) \xrightarrow{(-, d_2)} (-, M_1) \xrightarrow{(-, d_1)} (-, M_0) \xrightarrow{D_0} F \longrightarrow 0.$$

Evaluating the resolution at R yields a short exact sequence

$$0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow 0.$$

Since M_2 and M_1 are pure-injective (Proposition 5), they are both cotorsion. The module M_0 is therefore also cotorsion. ∎

The subcategory $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ of $((R\text{-mod})^{\text{op}}, \text{Ab})$ is closed under subobjects, products and extensions. It is therefore a torsion-free class. Denote by $\text{Gen}(-, R)$ the subcategory of $((R\text{-mod})^{\text{op}}, \text{Ab})$ consisting of all $(-, R)$-**generated** objects. The objects of $\text{Gen}(-, R)$ are the quotient objects of coproducts of copies of $(-, R)$. Since $(-, R)$ is a projective object, $\text{Gen}(-, R)$ is closed under extensions. It is therefore a torsion class in $((R\text{-mod})^{\text{op}}, \text{Ab})$. Denote by $t_R(F) \subseteq F$ the torsion subobject associated to F by the torsion class $\text{Gen}(-, R)$.

Proposition 13: *If the subcategory* $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab}) \subseteq ((R\text{-mod})^{\text{op}}, \text{Ab})$ *is regarded as the torsion-free class of a torsion theory, then* $\text{Gen}(-, R)$ *is the corresponding torsion class.*

Proof. Clearly, the object $(-, R)$ is torsion with respect to $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. It follows that every object of $\text{Gen}(-, R)$ is also torsion with respect to $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. To prove the reverse inclusion, let $F \in ((R\text{-mod})^{\text{op}}, \text{Ab})$ be an object that is torsion with respect to $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. There are no nonzero morphisms from $(-, R)$ to the quotient object $F/t_R(F)$, so it must be that $F/t_R(F) \in ((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ and so $F = t_R(F)$. ∎

The rule $F \mapsto \underline{F} := F/t_R(F)$ yields a functor from from $((R\text{-mod})^{\text{op}}, \text{Ab})$ to $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$. For example, if M is an R-module and $F = (-, M)$, then $\underline{F} \cong \underline{\text{Hom}}(-, \underline{M})$. Indeed, an epimorphism $g : R^{(\alpha)} \to M$ from a free module induces a flat presentation

$$(-, R)^{(\alpha)} \xrightarrow{(-, g)} (-, M) \longrightarrow \underline{\text{Hom}}(-, \underline{M}) \longrightarrow 0.$$

The sequence is exact, because if a morphism $f : A \to M$ with finitely presented domain A factors through a flat object, $f = hk$, where $k : A \to Z$ and Z is flat, then k factors through a finitely generated projective module, and hence so does f. But then f must factor through $g : R^{(\alpha)} \to M$. Now $\underline{\text{Hom}}(-, \underline{M})$ is torsion-free and the image of $(-, g)$ is torsion. Thus $\text{Im}(-, g) = t_R(F)$ and $\underline{F} = \underline{\text{Hom}}(-, \underline{M})$.

If $F \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and $G \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$, then, as with any torsion theory, there is an isomorphism of abelian groups

$$[\underline{F}, G] \cong [F, G],$$

natural in both F and G. In other words, the functor $F \mapsto \underline{F}$ is the left adjoint of the inclusion functor from $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ to $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. By [33, Proposition 9.4], it is right exact and preserves direct limits.

Proposition 14: *The functor* $F \mapsto \underline{F}$

$$\textit{from} \ \ ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab}) \quad \textit{to} \ \ ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$$

preserves representable objects, finitely presented objects and flat objects.

Proof. If A is a finitely presented module and $F = (-, A)$, then $\underline{F}$ is the representable object $\underline{\mathrm{Hom}}(-, \underline{A})$ of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. If $F \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is finitely presented, consider a presentation of F by representable objects

$$(-, A) \xrightarrow{(-, f)} (-, B) \longrightarrow F \longrightarrow 0.$$

Because the functor $F \mapsto \underline{F}$ is right exact, one obtains a presentation of $\underline{F}$ in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ by representable objects

$$\underline{\mathrm{Hom}}(-, \underline{A}) \xrightarrow{(-, \underline{f})} \underline{\mathrm{Hom}}(-, \underline{B}) \longrightarrow \underline{F} \longrightarrow 0.$$

If M is an arbitrary R-module, it is isomorphic to a direct limit $M = \varinjlim A_i$ of finitely presented modules A_i. The functor $\underline{F}$ preserves direct limits, so if we apply it to both sides of the equation $(-, M) = \varinjlim (-, A_i)$, we obtain that

$$\underline{\mathrm{Hom}}(-, \underline{M}) \cong \varinjlim \underline{\mathrm{Hom}}(-, \underline{A_i})$$

is flat. ∎

Benson and Gnacadja [10] give an example of a group ring $R = k[G]$ where not every flat object of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is of the form $\underline{\mathrm{Hom}}(-, \underline{M})$.

2.2. The functor $\mathrm{Ext}^1(-,M)$. Let M be an R-module. Since $\mathrm{Ext}^1(R,M)=0$, the contravariant functor $\mathrm{Ext}^1(-,M)$ is a functor on the stable category R-$\underline{\mathrm{mod}}$. A morphism $f : M \to N$ in R-Mod induces a morphism

$$\mathrm{Ext}^1(-,f) : \mathrm{Ext}^1(-,M) \to \mathrm{Ext}^1(-,N)$$

in the category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.

Proposition 15: *Let $f : M \to N$ be a monomorphism. Then f is a pure-monomorphism if and only if the induced morphism*

$$\mathrm{Ext}^1(-,f) : \mathrm{Ext}^1(-,M) \to \mathrm{Ext}^1(-,N)$$

is a monomorphism in the category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.

Proof. The short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} K \to 0$ induces a long exact sequence of functors

$$0 \longrightarrow (-,M) \xrightarrow{(-,f)} (-,N) \xrightarrow{(-,g)} (-,K) \xrightarrow{\Delta}$$

$$\longrightarrow \mathrm{Ext}^1(-,M) \xrightarrow{\mathrm{Ext}^1(-,f)} \mathrm{Ext}^1(-,N).$$

Then $\mathrm{Ext}^1(-,f)$ is a monomorphism if and only if $\Delta = 0$ if and only if $(-,g)$ is an epimorphism if and only if g is a pure-epimorphism if and only if f is a pure-monomorphism. ∎

A module X is **absolutely pure** if every monomorphism $f : X \to Y$ is a pure-monomorphism. Absolutely pure modules may be characterized as the modules X for which the pure-injective envelope $x : X \to \mathrm{PE}(X)$ is the injective envelope. Equivalently, $\mathrm{Ext}^1(A,X) = 0$ for every finitely presented left R-module A; it is for this reason that absolutely pure modules are also called **fp-injective.** We may express this by the equation

$$\mathrm{Ext}^1(-,X) = 0$$

in the category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

Let $e : M \to E$ be the injective envelope of a left R-module M; the cokernel is denoted by $\Omega^{-1}(M) := E/M$. The short exact sequence

$$0 \longrightarrow M \xrightarrow{e} E \xrightarrow{p} \Omega^{-1}(M) \longrightarrow 0$$

induces a long exact sequence of functors. Because $\mathrm{Ext}^1(-,E) = 0$, the first few terms of this long exact sequence are given by

$$0 \longrightarrow (-,M) \xrightarrow{(-,e)} (-,E) \xrightarrow{p} (-,\Omega^{-1}(M)) \xrightarrow{D} \mathrm{Ext}^1(-,M) \longrightarrow 0,$$

which is a flat resolution of $\mathrm{Ext}^1(-,M)$ in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. If M is pure-injective, the flat resolution is special, by Proposition 5.

Given R-modules M and N denote by $\mathrm{Abs}(M,N) \subseteq \mathrm{Hom}_R(M,N)$ the subgroup of morphisms $f : M \to N$ that factor through an fp-injective module. The preadditive category $R\text{-}\overline{\mathrm{Mod}}$ is defined to have the objects $\overline{M}$, where M is a left R-module, and the morphism groups given by

$$\overline{\mathrm{Hom}}(\overline{M}, \overline{N}) : \mathrm{Hom}_R(M,N)/\mathrm{Abs}(M,N).$$

Since $\mathrm{Ext}^1(-,X) = 0$ for every fp-injective module X, the functor

$$M \mapsto \mathrm{Ext}^1(-,M)$$

from R-Mod to $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ induces a functor

$$\mathrm{Ext}^1(-,?) : R\text{-}\overline{\mathrm{Mod}} \to ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$$

given by $\overline{M} \mapsto \mathrm{Ext}^1(-,M)$.

Theorem 16: *Let M be a left R-module and N a pure-injective left R-module. The induced morphism of abelian groups*

$$\mathrm{Ext}^1(-,?) : \overline{\mathrm{Hom}}(\overline{M}, \overline{N}) \to [\mathrm{Ext}^1(-,M), \mathrm{Ext}^1(-,N)],$$

defined by $\overline{f} \mapsto \mathrm{Ext}^1(-,f)$ is an isomorphism.

Proof. To prove that the morphism is onto, let $\eta : \mathrm{Ext}^1(-,M) \to \mathrm{Ext}^1(-,N)$ be a morphism in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and consider the respective flat resolutions of $\mathrm{Ext}^1(-,M)$ and $\mathrm{Ext}^1(-,N)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (-,M) & \longrightarrow & (-,E(M)) & \longrightarrow & (-,\Omega^{-1}(M)) & \longrightarrow & \mathrm{Ext}^1(-,M) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (-,f)} & & \Big\downarrow{\scriptstyle (-,g)} & & \Big\downarrow{\scriptstyle (-,k)} & & \Big\downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & (-,N) & \longrightarrow & (-,E(N)) & \longrightarrow & (-,\Omega^{-1}(N)) & \longrightarrow & \mathrm{Ext}^1(-,N) & \longrightarrow & 0
\end{array}$$

Because N is pure-injective, the flat resolution of $\mathrm{Ext}^1(-,N)$ is special. This gives rise to a morphism of flat resolutions, unique up to homotopy, indicated by the dotted arrows. Evaluating the commutative diagram at the module R yields a morphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{e_M} & E(M) & \xrightarrow{p_M} & \Omega^{-1}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & N & \xrightarrow{e_N} & E(N) & \xrightarrow{p_N} & \Omega^{-1}(N) & \longrightarrow & 0. \end{array}$$

Since $E(M)$ and $E(N)$ are injective, the induced morphism of long exact sequences of functors implies that $\eta = \mathrm{Ext}^1(-,f)$.

To demonstrate that the morphism of abelian groups is injective, suppose that a morphism of flat resolutions as above induces the morphism $\eta = \mathrm{Ext}^1(-,f) = 0$. It must then be homotopic to the zero morphism, which implies that $f : M \to N$ factors through the injective envelope $e_M : M \to E(M)$. ∎

Given R-modules M and N, the theorem helps us make sense of what it means to be a morphism $\alpha : \mathrm{Ext}^1(-,M) \to \mathrm{Ext}^1(-,N)$ in the category $(R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. If $n : N \to \mathrm{PE}(N)$ is the pure-injective envelope, then, according to the theorem, there is an R-morphism $f : M \to \mathrm{PE}(N)$ that makes the diagram

$$\begin{array}{ccc} & & \mathrm{Ext}^1(-,\mathrm{PE}(N)) \\ & \nearrow{\scriptstyle \mathrm{Ext}^1(-,f)} & \uparrow{\scriptstyle \mathrm{Ext}^1(-,n)} \\ \mathrm{Ext}^1(-,M) & \xrightarrow{\alpha} & \mathrm{Ext}^1(-,N) \end{array}$$

commute. Given a finitely presented module A and an extension $\zeta \in \mathrm{Ext}^1(A,M)$, the extension $\mathrm{Ext}^1(A,f)(\zeta)$ obtained by pushout along $f : M \to \mathrm{PE}(N)$ arises from the extension $\alpha(A)(\zeta) \in \mathrm{Ext}^1(A,N)$ by pushout along the pure-injective envelope of N. Since the morphism $\mathrm{Ext}^1(-,n)$ is a monomorphism, this property characterizes $\alpha(A)(\zeta)$.

2.3. Injective-free pure-injective modules. The proof of Theorem 16 shows that if a morphism $f : M \to N$, with N pure-injective, factors through an fp-injective module, then it factors through an injective module. This is not surprising, because if $f = gh$ where $g : K \to N$ and K is fp-injective, then g factors through the pure-injective envelope $k : K \to \mathrm{PE}(K)$, which is injective. Thus if M and N are pure-injective, then

$$\mathrm{Abs}(M, N) = \mathrm{Inj}(M, N),$$

where $\mathrm{Inj}(M, N) \subseteq \mathrm{Hom}_R(M, N)$ is the subgroup of morphisms that factor through an injective module. Let $R\text{-}\overline{\mathrm{Pinj}} \subseteq R\text{-}\overline{\mathrm{Mod}}$ be the subcategory of classes of the form $\overline{M}$ where M is pure-injective.

Call a pure-injective module M **injective-free** if it contains no nonzero injective summands.

Proposition 17: *Let M be a pure-injective left R-module. There is a direct sum decomposition*

$$M = M_{\mathrm{if}} \oplus E,$$

where E is injective and M_{if} is injective-free.

Proof. By Zorn's Lemma, there is a maximal fp-injective submodule $E \subseteq M$. As M is pure-injective, the pure-injective envelope $\mathrm{PE}(E)$ is an injective summand of M. By the choice of E, $E = \mathrm{PE}(E)$ is therefore injective. Thus

$$M = M_{\mathrm{if}} \oplus E,$$

where M_{if} is a pure-injective module with no nonzero fp-injective submodules. ∎

Let M be an injective-free pure-injective module. Then the injective envelope $e : M \to E$ is nowhere pure. The cokernel $p : E \to \Omega^{-1}(M)$ of the injective envelope is also nowhere pure, because it has an essential kernel. By Theorem 10, the flat resolution of $\mathrm{Ext}^1(-, M)$ given by

$$0 \longrightarrow (-, M) \xrightarrow{(-,e)} (-, E) \xrightarrow{(-,p)} (-, \Omega^{-1}(M)) \xrightarrow{D_0} \mathrm{Ext}^1(-, M) \longrightarrow 0$$

is minimal. If N is any pure-injective R-module, we may decompose it according to Proposition 17 as $N = N_{\mathrm{if}} \oplus E_N$, where N_{if} is an injective-free pure-injective, and E_N is injective. Then $\mathrm{Ext}^1(-, N) = \mathrm{Ext}^1(-, N_{\mathrm{if}})$, and a minimal flat resolution of $\mathrm{Ext}^1(-, N)$ is obtained by taking a flat resolution of $\mathrm{Ext}^1(-, N_{\mathrm{if}})$ as above.

Lemma 18: *Let ${}_RM$ be an injective-free pure-injective module and $S = \mathrm{End}_R M$. Then $\mathrm{Inj}(M, M) \subseteq J(S)$.*

Proof. The endomorphism ring S of a pure-injective module is an exchange ring, so if $\mathrm{Inj}(M, M) \not\subseteq J(\mathrm{End}_R M)$, then there is a nonzero idempotent $e \in \mathrm{Inj}(M, M)$. Write $e = fg$ where $g : M \to E$ and $f : E \to M$ with E injective. Then $e = (ef)(ge) = 1_{eM}$, where $ge : eM \to E$ and $fe : E \to eM$. But then eM is isomorphic to a nonzero summand of E, contradicting the assumption that M is injective-free. ∎

LEMMA 19: *Let $f : M \to N$ be an isomorphism between injective-free pure-injective modules such that $f : \overline{M} \to \overline{N}$ is an isomorphism in R-$\overline{\mathrm{Pinj}}$. Then f is an isomorphism.*

Proof. Let $g : N \to M$ be a morphism such that $\overline{f}\overline{g} = 1_{\overline{N}}$ and $\overline{g}\overline{f} = 1_{\overline{M}}$ in R-$\overline{\mathrm{Pinj}}$. By Lemma 18, $1_N - fg \in J(\mathrm{End}_R N)$ and $1_M - gf \in J(\mathrm{End}_R M)$. Thus $fg \in \mathrm{End}_R N$ and $gf \in \mathrm{End}_R M$ are both units, and so f is both injective and surjective. ∎

Let M and N be pure-injective R-modules and consider the respective direct sum decompositions, $M = M_{\mathrm{if}} \oplus E_M$ and $N = N_{\mathrm{if}} \oplus E_N$ given by Proposition 17. If $f : M \to N$ is an isomorphism, it may be represented with respect to these decompositions by a 2×2 matrix

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11} : M_{\mathrm{if}} \to N_{\mathrm{if}}$, $f_{12} : E_M \to N_{\mathrm{if}}$, $f_{21} : M_{\mathrm{if}} \to E_N$, and $f_{22} : E_M \to E_N$. In R-$\overline{\mathrm{Pinj}}$, $\overline{M} \cong \overline{M}_{\mathrm{if}}$ and $\overline{N} \cong \overline{N}_{\mathrm{if}}$ and $\overline{f} = \overline{f}_{11}$ is an isomorphism. By Lemma 19, $f_{11} : M_{\mathrm{if}} \to N_{\mathrm{if}}$ is an isomorphism. This justifies calling any direct summand of the form M_{if} ***the* injective-free part** of M.

Two pure-injective modules M and N are said to be **injectively stably isomorphic** if $\overline{M}$ and $\overline{N}$ are isomorphic in R-$\overline{\mathrm{Pinj}}$. Lemma 19 implies that M and N are injectively stably isomorphic if and only if they have isomorphic injective-free parts. Thus $M = M_{\mathrm{if}} \oplus E_M$ and $N = N_{\mathrm{if}} \oplus E_N$ with $M_{\mathrm{if}} \cong N_{\mathrm{if}}$. If $E = E_M^{\omega} \oplus E_N^{\omega}$, then, as in Eilenberg's trick, we have that

$$M \oplus E \;\cong\; M_{\mathrm{if}} \oplus E \;\cong\; N_{\mathrm{if}} \oplus E \;\cong\; N \oplus E.$$

On the other hand, if there is an injective module E, such that $M \oplus E \cong N \oplus E$, then M and N are unmistakably injectively stably isomorphic.

2.4. Injective objects. Every functor $F \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ admits a monomorphism into a functor of the form $\mathrm{Ext}^1(-, M)$. Indeed, a flat resolution of F of length 2 may instead be continued as a long exact sequence of functors

$$0 \longrightarrow (-, M) \xrightarrow{(-,f)} (-, N) \xrightarrow{(-,g)} (-, K) \xrightarrow{\Delta} \mathrm{Ext}^1(-, M) \longrightarrow \cdots .$$

This is possible because $g : N \to K$ is an epimorphism. As F is isomorphic to the image of the connecting morphism Δ, it may be embedded into $\mathrm{Ext}^1(-, M)$.

Proposition 20: *Every injective object in* $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ *is isomorphic to an object of the form* $\mathrm{Ext}^1(-, M)$, *where* M *is a pure-injective module.*

Proof. We may embed F into a functor of the form $\mathrm{Ext}^1(-, M)$, which in turn admits an embedding into $\mathrm{Ext}^1(-, \mathrm{PE}(M))$. Thus we may assume that F is isomorphic to a coproduct factor of $\mathrm{Ext}^1(-, M)$, where M is a pure-injective R-module. As F is injective, there is an idempotent endomorphism η of $\mathrm{Ext}^1(-, M)$ whose image is F. By Theorem 16, $\eta = \mathrm{Ext}^1(-, f)$ for some $f \in \mathrm{End}_R M$ which is idempotent modulo the two-sided ideal $\mathrm{Inj}(M, M)$. Since $\mathrm{End}_R M$ is an exchange ring, idempotents lift modulo any ideal [28, Corollary 1.3], so that there is an idempotent $e \in \mathrm{End}_R M$ such that $\eta = \mathrm{Ext}^1(-, e)$. It follows that $F \cong \mathrm{Ext}^1(-, eM)$. ∎

Given R-modules M and N, Yoneda's Lemma in R-Mod asserts the existence of an isomorphism of abelian groups,

$$\Theta_N : [(-, M), \mathrm{Ext}^1(-, N)]_{R\text{-Mod}} \cong \mathrm{Ext}^1(M, N),$$

natural in M, where $[(-, M), \mathrm{Ext}^1(-, N)]_{R\text{-Mod}}$ denotes the abelian group of natural transformations from $(-, M)$ to $\mathrm{Ext}^1(-, N)$ considered as functors on the category R-Mod of all R-modules. Denote by

$$\Delta_N(M) : \mathrm{Ext}^1(M, N) \to [(-, M), \mathrm{Ext}^1(-, N)]$$

the morphism obtained by composing Θ_N^{-1} with the morphism

$$[(-, M), \mathrm{Ext}^1(-, N)]_{R\text{-Mod}} \to [(-, M), \mathrm{Ext}^1(-, N)]$$

obtained by restriction to R-mod.

Theorem 21: *If* ${}_RN$ *is pure-injective, then for every module* M, *the morphism*

$$\Delta_N(M) : \mathrm{Ext}^1(M, N) \to [(-, M), \mathrm{Ext}^1(-, N)]$$

is an isomorphism.

Proof. M. Auslander [3, Proposition I.10.1] proved that a module N is pure-injective if and only if for every directed system $\{C_i\}_i$ of modules, the natural morphism

$$\mathrm{Ext}^1(\varinjlim C_i, N) \to \varprojlim \mathrm{Ext}^1(C_i, N)$$

is an isomorphism. Write $M = \varinjlim C_i$ as a direct limit of finitely presented modules C_i. Using Yoneda's Lemma, we have

$$\begin{aligned}\mathrm{Ext}^1(M,N) = \mathrm{Ext}^1(\varinjlim C_i, N) &\cong \varprojlim \mathrm{Ext}^1(C_i, N) \cong \varprojlim [(-,C_i), \mathrm{Ext}^1(-,N)] \\ &\cong [\varinjlim (-,C_i), \mathrm{Ext}^1(-,N)] \\ &\cong [(-,M), \mathrm{Ext}^1(-,N)]. \qquad \blacksquare\end{aligned}$$

The main consequence of Theorem 21 is an analogue of [6, Proposotion 3.2].

Proposition 22: *If ${}_RN$ is pure-injective, then the functor $\mathrm{Ext}^1_R(-,N)$ is an injective object of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. We will show that if $F \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$, then $\mathrm{Ext}^1[F, \mathrm{Ext}^1(-,N)] = 0$ in the ambient category $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$. Since the subcategory $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is closed under extensions, the same equation will hold there. Consider a resolution of F,

$$0 \longrightarrow (-,M_2) \xrightarrow{(-,d_2)} (-,M_1) \xrightarrow{(-,d_1)} (-,M_0) \xrightarrow{D_0} F \longrightarrow 0,$$

where $(-,M_0)$ and $(-,M_1)$ are projective. Applying the functor $[?, \mathrm{Ext}^1_R(-,N)]$ to this flat resolution yields the commutative diagram

$$\begin{array}{ccccc} [(-,M_0), \mathrm{Ext}^1(-,N)] & \longrightarrow & [(-,M_1), \mathrm{Ext}^1(-,N)] & \longrightarrow & [(-,M_2), \mathrm{Ext}^1(-,N)] \\ \uparrow{\scriptstyle \Delta_N(M_0)} & & \uparrow{\scriptstyle \Delta_N(M_1)} & & \uparrow{\scriptstyle \Delta_N(M_2)} \\ \mathrm{Ext}^1(M_0,N) & \xrightarrow{\mathrm{Ext}^1(d_1,N)} & \mathrm{Ext}^1(M_1,N) & \xrightarrow{\mathrm{Ext}^1(d_1,N)} & \mathrm{Ext}^1(M_2,N). \end{array}$$

Since $F(R) = 0$, the bottom row is exact. As Δ_N is an isomorphism, the top row is also exact. $\blacksquare$

These findings are summarized in the following

THEOREM 23: *The functor* $\mathrm{Ext}^1(-,?) : R\text{-}\overline{\mathrm{Pinj}} \to ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$, *given by* $\overline{N} \mapsto \mathrm{Ext}^1(-,N)$, *yields an equivalence of categories between* $R\text{-}\overline{\mathrm{Pinj}}$ *and the subcategory of injective objects of* $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

That these two categories are equivalent is a special case of a result [24, Theorem 5.3] due to H. Krause.

2.5. THE TORSION CLASS $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. The category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is a subcategory of $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$ closed under subobjects, quotient objects, coproducts and extensions. It therefore constitutes a hereditary torsion class. If $F \in ((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$, we will denote by $t(F)$ the torsion subobject of F corresponding to the torsion class $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

PROPOSITION 24: *Let* $I \in ((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$ *be an injective object. There is an injective-free pure-injective module* M *(unique up to isomorphism) such that*

$$I = E(\mathrm{Ext}^1(-,M)) \amalg I_0,$$

where $E(\mathrm{Ext}^1(-,M))$ *denotes the injective envelope of* $\mathrm{Ext}^1(-,M)$ *in* $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$ *and* I_0 *is a torsion-free injective object. The decomposition is unique up to isomorphism.*

Proof. The torsion subobject $t(I)$ of I has no essential extensions in the category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. It is therefore injective as an object of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. By Proposition 20, there is a pure-injective module M, which we may assume to be injective-free, such that $t(I) \cong \mathrm{Ext}^1(-,M)$. Then $I = E(\mathrm{Ext}^1(-,M)) \amalg I_0$, where I_0 is a torsion-free injective object of $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$. If another such decomposition $I = E(\mathrm{Ext}^1(-,M')) \amalg I'_0$ is given, then $\mathrm{Ext}^1(-,M') \cong t(I) \cong \mathrm{Ext}^1(-,M)$. Since M and M' are both injective-free pure-injective modules, Theorem 23 implies that $M \cong M'$. The identity morphism 1_I induces an isomorphism from $E(\mathrm{Ext}^1(-,M))$ to $E(\mathrm{Ext}^1(-,M'))$. By [33, Lemma V.5.3], the torsion-free parts I_0 and I'_0 are also isomorphic. ∎

If M is an R-module, then the flat object $(-,M)$ is torsion-free. For, suppose that G belongs to $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. There is a projective presentation

$$(-,N) \xrightarrow{(-,f)} (-,K) \longrightarrow G \longrightarrow 0,$$

where $f : N \to K$ is an epimorphism. Applying the functor $[?, (-,M)]$ yields an exact sequence of abelian groups

$$0 \longrightarrow [G,(-,M)] \longrightarrow (K,M) \xrightarrow{(f,M)} (N,K).$$

Since f is an epimorphism, $(f,M):(K,M)\to(N,K)$ is a monomorphism of groups. Thus $[G,(-,M)]=0$, and $(-,M)$ is torsion-free.

Proposition 25: *If $(\mathcal{T},\mathcal{F})$ is the torsion theory whose torsion class is given by $\mathcal{T}=((R\text{-}\underline{\text{mod}})^{\text{op}},\text{Ab})$, then the class $\mathcal{F}$ of torsion-free objects consists of the objects of flat dimension at most* 1.

Proof. Recall that a functor $F\in((R\text{-mod})^{\text{op}},\text{Ab})$ is of flat dimension at most 1 if and only if it is isomorphic to a subfunctor of a flat functor. Thus every functor of flat dimension at most 1 is torsion-free. Conversely, suppose that G is a torsion-free object with projective presentation

$$\begin{array}{ccccccc}
(-,N) & \xrightarrow{(-,f)} & (-,K) & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (-,g)} & \swarrow{\scriptstyle \mu} & & & \\
 & & (-,L) & & & &
\end{array}$$

If $g:K\to L$ denotes the cokernel of $f:N\to K$, then there is unique morphism $\mu:G\to(-,L)$ as indicated by the dotted arrow. Evaluating the diagram at the module R, shows that $\mu(R)$ is a monomorphism, and hence that $\operatorname{Ker}\mu$ belongs to $((R\text{-}\underline{\text{mod}})^{\text{op}},\text{Ab})$. But G is torsion-free, so it must be that $\operatorname{Ker}\mu=0$, and hence that G admits an embedding into a flat object. ∎

Theorem 26: *Let $I\in((R\text{-mod})^{\text{op}},\text{Ab})$ be an injective object. There is an injective-free pure-injective module M and an injective module E such that*

$$I=E(\text{Ext}^1(-,M))\amalg(-,E).$$

If $I=E(\text{Ext}^1(-,M'))\amalg(-,E')$ is another such decomposition, then $M'\cong M$ and $E'\cong E$.

Proof. By Proposition 24, it suffices to prove that a torsion-free injective object I of $((R\text{-mod})^{\text{op}},\text{Ab})$ must be of the form $I\cong(-,E)$, where E is an injective module. By Proposition 25, there is an embedding of I into a flat functor $(-,L)$. The embedding is necessarily a split monomorphism, so that there is a direct summand E of L, such that $I \cong (-, E)$. Obviously, E must be an injective R-module. ∎

The proof of Theorem 26 indicates that the torsion theory whose torsion class is $\mathcal{T} = ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is cogenerated by the injective objects of the form $(-, E)$, as E ranges over the class of injective R-modules.

3. The category of covariant functors.

The category of **covariant** additive functors $F : \text{mod-}R \to \mathrm{Ab}$ is denoted by $(\text{mod-}R, \mathrm{Ab})$. Like the category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, it is a locally finitely presented Grothendieck category, and, as in the case of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, there is a full and faithful functor from R-Mod to $(\text{mod-}R, \mathrm{Ab})$. It is given by the rule $M \mapsto - \otimes_R M$, which is a right exact functor, in contrast to the full and faithful functor $M \mapsto (-, M)$ into the category of contravariant functors, which is left exact. It is because of our preference for left R-modules and these natural embeddings of R-Mod into $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and $(\text{mod-}R, \mathrm{Ab})$, respectively, that we choose R-mod as our domain for the contravariant functors and mod-R as the appropriate domain for the category of covariant functors.

3.1. The Auslander-Bridger Transpose. The stable category $\underline{\mathrm{mod}}\text{-}R$ of the category of finitely presented right R-modules is defined as for the category R-mod. The functor category $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ may be identified with the subcategory of $(\text{mod-}R, \mathrm{Ab})$ of functors F for which $F(R_R) = 0$. The Auslander-Bridger Transpose [5] is an equivalence of categories $\mathrm{Tr} : \underline{\mathrm{mod}}\text{-}R \to (R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}$. It is defined as follows: let A_R be a finitely presented right R-module with a presentation

$$R^n_R \xrightarrow{f} R^m_R \xrightarrow{p} A_R \to 0 \tag{2}$$

by free modules of finite rank. Applying the left exact contravariant functor $(-)^* = (-, R_R)$ yields an exact sequence

$$0 \longrightarrow A^* \xrightarrow{p^*} R^m \xrightarrow{f^*} R^n \longrightarrow \mathrm{Tr}(A) \longrightarrow 0 \tag{3}$$

of left R-modules, where $\mathrm{Tr}(A)$ is the cokernel of f^*. The object $\mathrm{Tr}(A)$ is not well-defined, but its class in R-$\underline{\mathrm{mod}}$ is. The inverse of Tr is defined similarly and also denoted by $\mathrm{Tr} : (R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}} \to \underline{\mathrm{mod}}\text{-}R$. The Auslander-Bridger Transpose induces an equivalence of functor categories $\mathrm{Tr}_* : ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}) \to (\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$, given by $\mathrm{Tr}_* : F \mapsto F \circ \mathrm{Tr}$. The category of contravariant functors on the stable category R-$\underline{\mathrm{mod}}$ is therefore equivalent to the category of covariant functors on the stable category $\underline{\mathrm{mod}}$-$R = R^{\mathrm{op}}$-$\underline{\mathrm{mod}}$.

Just as the contravariant functors of the form $(-, M)$ are characterized as the flat objects of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$, the covariant functors of the form $- \otimes_R M$ may be intrinsically characterized as the fp-injective objects of $(\text{mod-}R, \mathrm{Ab})$ (cf. [13, Lemma 1.4]). Recall that an object F is **fp-injective** if $\mathrm{Ext}^1(A, F) = 0$ for every finitely presented object $A \in (\text{mod-}R, \mathrm{Ab})$. The relationship between the functors $- \otimes M$ and $(-, M)$ is clarified by the following. The argument is dual to the proof of [7, Proposition 2.2].

Proposition 27: *For every R-module M, there is an exact sequence in* $(\text{mod-}R, \mathrm{Ab})$,

$$0 \longrightarrow \mathrm{Tr}_*[\mathrm{Ext}^1(-, M)] \longrightarrow - \otimes_R M \xrightarrow{\alpha_M} ((-)^*, M)$$

natural in M.

Proof. Let A_R be a finitely presented right R-module with the free presentation (2) given above. Applying the right exact functor $- \otimes_R M$ to this free presentation, and the left exact functor $(-, M)$ to the exact sequence (3), yields the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & R^n \otimes_R M & \xrightarrow{f \otimes M} & R^m \otimes_R M & \xrightarrow{p \otimes M} & A \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \alpha_M(A) & & \\
0 \longrightarrow & (\mathrm{Tr}(A), M) \longrightarrow & (R^n, M) & \xrightarrow{(f^*, M)} & (R^m, M) & \xrightarrow{(p^*, M)} & (A^*, M). & &
\end{array}$$

The first two vertical arrows are isomorphisms as indicated. The top row is exact, so that $A \otimes_R M$ is the cokernel of $f \otimes M$. The morphism $\alpha_M(A) : A \otimes_R M \to (A^*, M)$ is induced by the universal property of the cokernel and the fact that $(p^*, M)(f^*, M) = 0$. It is functorial in A and because

$$\mathrm{Ext}^1(\mathrm{Tr}(A), M) = \mathrm{Ker}\,(p^*, M)/\mathrm{Im}(f^*, M),$$

$\mathrm{Ker}\,\alpha_M$ is naturally isomorphic to $\mathrm{Ext}^1(\mathrm{Tr}(-), M) = \mathrm{Tr}_*[\mathrm{Ext}^1(-, M)]$. ∎

It may be shown that a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0$$

is pure-exact provided the corresponding sequence

$$0 \longrightarrow - \otimes_R M \xrightarrow{- \otimes f} - \otimes_R N \xrightarrow{- \otimes g} - \otimes_R K \longrightarrow 0$$

is exact in (mod-R, Ab). Indeed, if $f : M \to N$ is a monomorphism, then the naturality of Proposition 27 yields a morphism of exact sequences as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tr}_*[\mathrm{Ext}^1(-, M)] & \longrightarrow & - \otimes M & \xrightarrow{\alpha_M} & ((-)^*, M) \\ & & \downarrow{\scriptstyle \mathrm{Tr}_*[\mathrm{Ext}^1(-,f)]} & & \downarrow{\scriptstyle - \otimes f} & & \downarrow{\scriptstyle ((-)^*, f)} \\ 0 & \longrightarrow & \mathrm{Tr}_*[\mathrm{Ext}^1(-, N)] & \longrightarrow & - \otimes N & \xrightarrow{\alpha_M} & ((-)^*, N). \end{array}$$

Because $((-)^*, f) : ((-)^*, M) \to ((-)^*, N)$ is a monomorphism, we see that $- \otimes f$ is a monomorphism if and only if $\mathrm{Tr}_*[\mathrm{Ext}^1(-, f)]$ is. By Proposition 15, this is equivalent to f being a pure-monomorphism.

If M is pure-injective, then the functor $- \otimes_R M$ is an injective object of (mod-R, Ab). Indeed, the injective objects of (mod-R, Ab) have been characterized by Gruson and Jensen [17] as the functors isomorphic to those of the form $- \otimes_R M$, where M is a pure-injective module. Given $F \in$ (mod-R, Ab), there is an injective copresentation

$$0 \longrightarrow F \longrightarrow - \otimes_R M \xrightarrow{\eta} - \otimes_R N.$$

Because the embedding $M \mapsto - \otimes_R M$ is full, there is a morphism $f : M \to N$ of R-modules such that $\eta = - \otimes f$. If $K = \mathrm{Coker}\, f$ is the cokernel of f, then the right exactness of the tensor functor permits one to complete the copresentation to a coresolution of F by fp-injective objects as follows

$$0 \longrightarrow F \longrightarrow - \otimes_R M \xrightarrow{- \otimes f} - \otimes_R N \longrightarrow - \otimes K \longrightarrow 0.$$

As in the contravariant case, the subcategory ($\underline{\mathrm{mod}}$-R, Ab) is a hereditary torsion class of (mod-R, Ab). If $F \in$ (mod-R, Ab), we will denote the ($\underline{\mathrm{mod}}$-R, Ab)-torsion subobject of F also by $t(F)$. Let us note that if E is an injective R-module, then $- \otimes_R E$ is a torsion-free object of (mod-R, Ab). For, suppose that $F \in$ ($\underline{\mathrm{mod}}$-R, Ab) and consider an injective copresentation of F,

$$0 \longrightarrow F \longrightarrow - \otimes_R M_0 \xrightarrow{- \otimes f} - \otimes_R M_1.$$

Because $F(R) = 0$, the morphism $f : M_0 \to M_1$ of pure-injective modules is a monomorphism. Applying the functor $[?, - \otimes_R E]$ yields an exact sequence of abelian groups. The embedding $M \mapsto - \otimes_R M$ is full, so the sequence is isomorphic to

$$(M_1, E) \xrightarrow{(f,E)} (M_0, E) \longrightarrow [F, - \otimes E] \longrightarrow 0.$$

Since f is a monomorphism and E is injective, the morphism

$$(f, E) : (M_1, E) \to (M_0, E)$$

of abelian groups is an epimorphism. By the exactness of the sequence, $[F, - \otimes E] = 0$. These considerations may be used to determine the torsion subobject of the functor $- \otimes_R M$.

PROPOSITION 28: *Let $e : M \to E$ be the injective envelope of M. The torsion subobject of $- \otimes M$ is part of the exact sequence*

$$0 \longrightarrow t(- \otimes M) \longrightarrow - \otimes M \xrightarrow{- \otimes e} - \otimes E.$$

Proof. The kernel of $- \otimes e$ is torsion, while its image is a subobject of $- \otimes E$, and is thus torsion-free. ∎

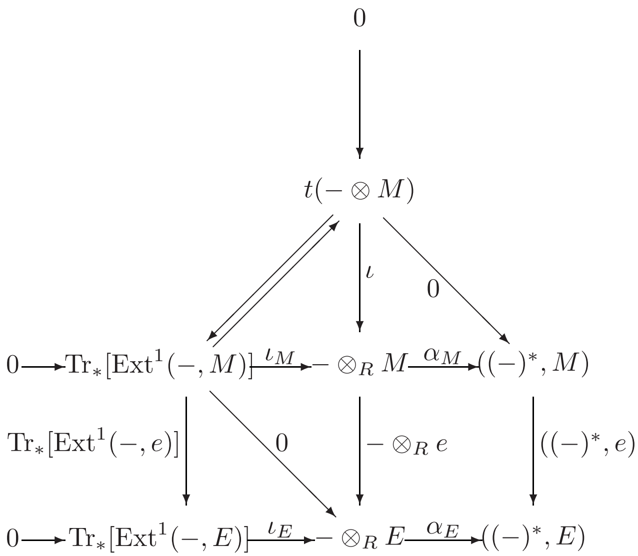
By Theorem 23, the injective objects of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ are of the form $\text{Ext}^1(-, M)$, where M is pure-injective. Because the injective objects of $(\text{mod-}R, \text{Ab})$ are of the form $- \otimes_R M$, where M is injective, the injective objects of $(\underline{\text{mod}}\text{-}R, \text{Ab})$ are the objects of the form $t(- \otimes_R M)$, M injective. This is because these are precisely the objects with no essential extensions in $(\underline{\text{mod}}\text{-}R, \text{Ab})$. It is therefore no surprise that if M is pure-injective, then the injective object $\text{Ext}^1(-, M)$ of $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is associated by Tr_* to the injective object $t(- \otimes M)$ of $(\underline{\text{mod}}\text{-}R, \text{Ab})$. More generally, one has the following. The argument is dual to the proof of [7, Proposition 2.2].

THEOREM 29: *Given an R-module M, there is an isomorphism*

$$\text{Tr}_*[\text{Ext}^1(-, M)] \cong t(- \otimes M),$$

natural in M.

Proof. Let $e : M \to E$ be the pure-injective envelope of M. We obtain the following commutative diagram



where the rows and column are exact sequences; the morphism

$$\alpha_M : - \otimes M \to ((-)^*, M)$$

is given by Proposition 27. The composition $(- \otimes e)\iota_M = 0$, because it factors through $\mathrm{Tr}_*[\mathrm{Ext}^1(-, E)] = 0$. Also, the injective envelope $e : M \to E$ is a monomorphism, so that $((-)^*, e) : ((-)^*, M) \to ((-)^*, E)$ is a monomorphism of functors. But $((-)^*, e)\alpha_M \iota = \alpha_E(- \otimes e)\iota = 0$ which implies that $\alpha_M \iota = 0$. The morphisms between $\mathrm{Tr}_*[\mathrm{Ext}^1(-, M)]$ and $t(- \otimes_R M)$ are then induced by the universal property of a kernel. They are both monomorphisms, so that $\mathrm{Tr}_*[\mathrm{Ext}^1(-, M)]$ and $t(- \otimes M)$ represent the same subobject of $- \otimes M$. Naturality may be verified using a routine argument. ∎

By Theorem 29, an injective object $-\otimes M$ is torsion-free if and only if M is injective. Just as the hereditary torsion class $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ of $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$ was cogenerated by the injective objects of the form $(-, E)$, where E is an injective R-module, this shows that the hereditary torsion class $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ of $(\mathrm{mod}\text{-}R, \mathrm{Ab})$ is cogenerated by the injective objects of the form $- \otimes_R E$, where E is an injective R-module. Theorem 29 may also be used to give an immediate proof of Propositions 20 and 22. Naturality may be used to give an alternative proof of Theorem 23.

Notice that if E is an injective R-module, then a morphism $f : E \to N$ is nowhere pure if and only if the kernel of f is essential in E.

PROPOSITION 30: *Let M be a pure-injective module. A morphism $f : M \to N$ is nowhere pure if and only if* $\mathrm{Ker}\,(- \otimes f) \subseteq - \otimes M$ *is an essential subfunctor.*

Proof. The functor $M \mapsto - \otimes M$ is full, so that the endomorphism rings $S = \mathrm{End}_R M$ and $\mathrm{End}(- \otimes M)$ may be identified. Since M is pure-injective, the object $- \otimes M$ is injective in the category $(\text{mod-}R, \mathrm{Ab})$. An endomorphism $h : M \to M$ thus belongs to the Jacobson radical of S if and only if $\mathrm{Ker}\,(- \otimes h)$ is essential in $- \otimes M$. The proposition then follows from Proposition 9. ∎

Let us describe how the equivalence $\mathrm{Tr}_* : ((\underline{\text{mod}}\text{-}R, \mathrm{Ab}) \to ((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab})$ acts on a general object $G \in ((\underline{\text{mod}}\text{-}R, \mathrm{Ab})$. Consider a copresentation of G by fp-injective objects in $(\text{mod-}R, \mathrm{Ab})$,

$$0 \longrightarrow G \longrightarrow - \otimes M \xrightarrow{\ - \otimes f\ } - \otimes N.$$

Because the torsion class $((\underline{\text{mod}}\text{-}R, \mathrm{Ab})$ is hereditary, the torsion functor $t(-)$ is left exact. Since $t(G) = G$, applying the functor t yields the exact sequence

$$0 \longrightarrow G \longrightarrow t(- \otimes M) \xrightarrow{\ t(- \otimes f)\ } t(- \otimes N).$$

Applying the equivalence Tr_* yields an exact sequence

$$0 \longrightarrow \mathrm{Tr}_*(G) \longrightarrow \mathrm{Ext}^1(-, M) \xrightarrow{\ \mathrm{Ext}^1(-, f)\ } \mathrm{Ext}^1(-, N)$$

in the category $((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab})$. Since G belongs to $(\underline{\text{mod}}\text{-}R, \mathrm{Ab})$, the morphism $f : M \to N$ is a monomorphism. Let $g : N \to K$ be the cokernel of f. It follows that $\mathrm{Tr}_*(G)$ is the object of $((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab})$ given by the flat resolution

$$0 \longrightarrow (-, M) \xrightarrow{(-, f)} (-, N) \xrightarrow{(-, g)} (-, K) \longrightarrow \mathrm{Tr}_*(G) \longrightarrow 0.$$

If the copresentation of G is a minimal injective copresentation in $(\text{mod-}R, \mathrm{Ab})$, then the fp-injective coresolution

$$0 \longrightarrow G \longrightarrow - \otimes M \xrightarrow{- \otimes f} - \otimes N \xrightarrow{- \otimes g} - \otimes K \longrightarrow 0$$

of G has the property that M and N are pure-injective and the kernels of both $- \otimes f$ and $- \otimes g$ are essential in $- \otimes M$ and $- \otimes N$, respectively. By Proposition 30, the corresponding flat resolution of $\mathrm{Tr}_*(G)$ is minimal.

3.2. Injective envelopes. Recall from [17] that if $m : M \to \mathrm{PE}(M)$ is the pure-injective envelope of M, then the morphism $- \otimes m : - \otimes M \to - \otimes \mathrm{PE}(M)$ is the injective envelope of $- \otimes M$ in the category (mod-R, Ab).

Theorem 31: *Let $m : M \to \mathrm{PE}(M)$ be the pure-injective envelope of M. Then $\mathrm{Ext}^1(-, m) : \mathrm{Ext}^1(-, M) \to \mathrm{Ext}^1(-, \mathrm{PE}(M))$ is the injective envelope of $\mathrm{Ext}^1(-, M)$ in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. The pure-injective envelope $m : M \to \mathrm{PE}(M)$ is a pure-monomorphism, so the induced morphism $-\otimes m : (-\otimes M) \to (-\otimes \mathrm{PE}(M))$ is a monomorphism of covariant functors. Since the torsion theory whose torsion class is $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ is hereditary, the restriction to the torsion subobjects is also a monomorphism

$$t(- \otimes m) : t(- \otimes M) \to t(- \otimes \mathrm{PE}(M)).$$

Its image is essential, because $- \otimes M$ is essential in $- \otimes \mathrm{PE}(M)$. By Theorem 29, this is just the morphism $\mathrm{Ext}^1(-, m) : \mathrm{Ext}^1(-, M) \to \mathrm{Ext}^1(-, \mathrm{PE}(M))$. ∎

The next result will be used to relate the minimal flat resolution in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of an object $G \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ and its minimal injective copresentation in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

Theorem 32: *Let $f : M \to N$ be a morphism of R-modules with M pure-injective. The kernel of the induced morphism*

$$\mathrm{Ext}^1(-, f) : \mathrm{Ext}^1(-, M) \to \mathrm{Ext}^1(-, N)$$

is essential in $\mathrm{Ext}^1(-, M)$ if and only if the restriction of f to the injective-free part of M is nowhere pure.

Proof. Let $f_{\mathrm{if}} : M_{\mathrm{if}} \to N$ be the restriction of f to the injective-free part of M. Because f and f_{if} induce the same morphism on $\mathrm{Ext}^1(-, M_{\mathrm{if}}) = \mathrm{Ext}^1(-, M)$, we may assume without loss of generality that $M = M_{\mathrm{if}}$ is injective-free.

Suppose that $f : M \to N$ is nowhere pure. If the kernel of $\mathrm{Ext}^1(-, f)$ is not essential in $\mathrm{Ext}^1(-, M)$, then there is a morphism $\eta : \mathrm{Ext}^1(-, N) \to \mathrm{Ext}^1(-, M)$ such that the composition $\eta \mathrm{Ext}^1(-, f)$ is a nonzero idempotent endomorphism of $\mathrm{Ext}^1(-, M)$. By Theorem 16, there is a morphism $g : N \to M$ such that

$\eta = \mathrm{Ext}^1(-,g)$. Thus gf is idempotent modulo $\mathrm{Abs}(M,M)$. Since $S = \mathrm{End}_R M$ is an exchange ring, idempotents lift modulo any two-sided ideal [28, Corollary 1.3], and there is a nonzero idempotent $e \in S$ such that $gf - e \in \mathrm{Abs}(M,M) \subseteq J(S)$, by Lemma 18. But then $gf \notin J(S)$, which contradicts Proposition 9.

If $f : M \to N$ is not nowhere pure, then there is a morphism $g : N \to M$ such that $gf = e \in \mathrm{End}_R M$ is a nonzero idempotent. The image of $\mathrm{Ext}^1(-,g)\mathrm{Ext}^1(-,f) = \mathrm{Ext}^1(-,e)$ is $\mathrm{Ext}^1(-,eM)$, which is a nonzero summand of $\mathrm{Ext}^1(-,M)$, because eM is not injective. The kernel of $\mathrm{Ext}^1(-,f)$ is then contained in the kernel of $\mathrm{Ext}^1(-,e)$, which is not essential in $\mathrm{Ext}^1(-,M)$. ∎

The following is an analogue of a result of Auslander and Reiten [6, Proposition 4.6].

Corollary 33: *Let $F \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ and suppose that a minimal flat resolution of F in $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$ is given by*

$$0 \longrightarrow (-,M_2) \xrightarrow{(-,d_2)} (-,M_1) \xrightarrow{(-,d_1)} (-,M_0) \xrightarrow{D_0} F \longrightarrow 0.$$

Then the minimal injective copresentation of F in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is given by

$$0 \longrightarrow F \longrightarrow \mathrm{Ext}^1(-,M_2) \xrightarrow{\mathrm{Ext}^1(-,d_2)} \mathrm{Ext}^1(-,M_1).$$

Proof. Evaluate the minimal flat resolution of F at R to obtain the short exact sequence

$$0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow 0.$$

By Proposition 5 and Theorem 10, M_2 and M_1 are pure-injective and d_2 and d_1 are nowhere pure. The beginning of the associated long exact sequence of functors is given by

$$0 \longrightarrow (-,M_2) \xrightarrow{(-,d_2)} (-,M_1) \xrightarrow{(-,d_1)} (-,M_0) \xrightarrow{\Delta_0} \mathrm{Ext}^1(-,M_2) \xrightarrow{\mathrm{Ext}^1(-,d_2)} \cdots .$$

The functor F is the image of the first connecting morphism Δ_0. By exactness, it is the kernel of $\mathrm{Ext}^1(-,d_2)$, which is essential, by Theorem 32, since M_1 is injective-free. Thus $\mathrm{Ext}^1(-,M_2)$ is the injective envelope of F. Since d_1 is nowhere pure on M_1, its restriction to the injective-free part of M_1 is also nowhere pure. Theorem 32 then implies that the kernel of $\mathrm{Ext}^1(-, d_1)$ is essential in $\mathrm{Ext}^1(-, M_1)$. By exactness, $\mathrm{Ext}^1(-, M_1)$ is the injective envelope of the image of $\mathrm{Ext}^1(-, d_2)$. Thus the injective copresentation is minimal. ∎

Next, we prove a converse to Corollary 33. It is a consequence of Theorem 32, whose proof depends on the observation that if $f : M \to K$ and $g : N \to K$ are nowhere pure maps, with M and N pure-injective, then so is the coproduct $f \amalg g : M \oplus N \to K$. This is because the kernel of $- \otimes (f \amalg g)$ contains $\mathrm{Ker}\,(- \otimes f) \amalg \mathrm{Ker}\,(- \otimes g)$, which, by Proposition 30, is essential in $(- \otimes M) \amalg (- \otimes N) = - \otimes (M \oplus N)$.

Corollary 34: *Suppose that the minimal injective resolution of* $F \in ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ *is given by*

$$0 \longrightarrow F \longrightarrow \mathrm{Ext}^1(-, M) \xrightarrow{\mathrm{Ext}^1(-, f)} \mathrm{Ext}^1(-, N),$$

where $f : M \to N$ *and* M *and* N *are injective-free pure-injective modules. Let* $g' : \mathrm{Ker}\, f \to E$ *be the injective envelope of* $\mathrm{Ker}\, f$, *and* $g : M \to E$ *some extension of* g' *to* M. *The minimal flat resolution of* F *in* $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ *is induced by the short exact sequence*

$$0 \longrightarrow M \xrightarrow{(f,g)} N \oplus E \xrightarrow{h} K \longrightarrow 0$$

Proof. Considering the long exact sequence of functors induced by the short exact sequence shows that the first few terms give a flat resolution of F in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. The modules M and $N \oplus E$ are both pure-injective, so by Proposition 5, the flat resolution is special. By Theorem 10, it suffices to prove that (f, g) and h are both nowhere pure. To see that (f, g) is nowhere pure, suppose that there existed a morphism $s : N \oplus E$ such that $sf + sg = s(f, g) \notin J(\mathrm{End}_R M)$. By Theorem 32, $f : M \to N$ is nowhere pure, so that $sf \in J(\mathrm{End}_R M)$; it must be that $sg \notin J(\mathrm{End}_R M)$. Then $g : M \to E$ is not nowhere pure and we could find an $s' : E \to M$ such that $e = s'g \in \mathrm{End}_R M$ is a nonzero idempotent. Then eM is a summand of M isomorphic to a summand of E, contradicting the assumption that M was injective-free.

To see that $h : N \oplus E \to K$ is nowhere pure, it suffices to show that the restrictions of h to N and E, respectively, are nowhere pure. The restriction of h to the injective-free pure-injective module N is nowhere pure, because the kernel of $\mathrm{Ext}^1(-, h)$ is equal to the image of $\mathrm{Ext}^1(-, f)$, and is therefore essential in $\mathrm{Ext}^1(-,N)$. The restriction of h to E is also nowhere pure, because it has an essential kernel. This follows from the fact that $\mathrm{Ker}\, f$ considered as an essential submodule of E is contained in the image of (f,g), which is equal to the kernel of h. ∎

3.3. THE GABRIEL SPECTRUM. Let $\mathcal{C}$ be a Grothendieck category. The **Gabriel spectrum** of $\mathcal{C}$, denoted by $\mathrm{Sp}(\mathcal{C})$, is the set whose points are the indecomposable injective objects E — up to isomorphism — of the category $\mathcal{C}$. That this collection indeed forms a set follows from the fact that every indecomposable injective object is the injective envelope of a quotient object of the generator; up to isomorphism, there are only set-many such quotient objects. The Gabriel spectrum is topologized by defining a subset $\mathcal{O}$ to be *open* if there is a hereditary torsion class $\mathcal{T} \subseteq \mathcal{C}$ such that

$$\mathcal{O} = \mathcal{O}(\mathcal{T}) := \{E \in \mathrm{Sp}(\mathcal{C}) : \ E \text{ is not } \mathcal{T}\text{-torsion-free}\}.$$

We will call $\mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ the **contravariant Gabriel spectrum** of the ring R and $\mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$ the **covariant Gabriel spectrum** of R. A description of the points of the covariant Gabriel spectrum of R follows from the characterization [17] of the injective objects of $(\text{mod-}R, \mathrm{Ab})$ as those of the form $- \otimes_R M$, where M is a pure-injective R-module. The points of $\mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$ are therefore the isomorphism types of the functors $- \otimes_R U$, where U is an indecomposable pure-injective R-module. A similar description of the points of the contravariant Gabriel spectrum of R follows from Theorem 26.

COROLLARY 35: *The following is a list, complete and without repetition, of the injective indecomposable objects of* $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$:

(1) $E(\mathrm{Ext}^1(-,M))$, *the injective envelope of* $\mathrm{Ext}^1(-,M)$, *where M is a non-injective indecomposable pure-injective R-module;*

(2) $(-,E)$, *where E is an indecomposable injective R-module.*

The corollary implies that the points of $\mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$ and $\mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ may be placed in bijective correspondence, according to the rule

$$\Xi_R : - \otimes_R U \mapsto \begin{cases} E[\mathrm{Ext}^1(-,U)], & \text{if } U \text{ is not injective;} \\ (-,U), & \text{if } U \text{ is injective.} \end{cases} \tag{4}$$

A hereditary torsion class $\mathcal{T} \subseteq \mathcal{C}$ is itself a Grothendieck category and so has a Gabriel spectrum associated to it. To every indecomposable injective

$I \in \mathrm{Sp}(\mathcal{T})$ in the category $\mathcal{T}$, we may associate its injective envelope $E_{\mathcal{C}}(I)$ in $\mathcal{C}$, which is an indecomposable object, because I is uniform in $\mathcal{C}$. The association $I \mapsto E_{\mathcal{C}}(I)$ yields a continuous function $\mathrm{Sp}(E_{\mathcal{C}}) : \mathrm{Sp}(\mathcal{T}) \to \mathrm{Sp}(\mathcal{C})$ that consitutes a homeomorphism from $\mathrm{Sp}(\mathcal{T})$ to its image, the open subset $\mathcal{O}(\mathcal{T})$.

For example, the functor category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is a hereditary torsion class in $((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab})$. The Gabriel spectrum $\mathrm{Sp}((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ of the contraviant functors on the stable category is therefore homeomorphic to the open subset $\mathcal{O}((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ of the contravariant Gabriel spectrum of R. Similarly, the functor category $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ is a hereditary torsion class in $(\mathrm{mod}\text{-}R, \mathrm{Ab})$, and so the Gabriel spectrum $\mathrm{Sp}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ is homeomorphic to the open subset $\mathcal{O}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ of the covariant Gabriel spectrum of R. The equivalence of categories $\mathrm{Tr}_* : ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}) \to (\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ induced by the Auslander-Bridger Transpose yields a homeomorphism $\mathrm{Sp}(\mathrm{Tr}_*)$ of the respective Gabriel spectra of stable functor categories. It is part of the commutative diagram

$$\begin{array}{ccc}
\mathrm{Sp}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab}) & \xrightarrow{\mathrm{Sp}(E)} & \mathrm{Sp}(\mathrm{mod}\text{-}R, \mathrm{Ab}) \\
{\scriptstyle \mathrm{Sp}(\mathrm{Tr}_*)}\downarrow & & \downarrow{\scriptstyle \Xi_R} \\
\mathrm{Sp}((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\mathrm{Sp}(E)} & \mathrm{Sp}((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}),
\end{array}$$

where the morphisms $\mathrm{Sp}(E)$ are induced by taking injective envelopes in the respective ambient categories.

Associated to a hereditary torsion class $\mathcal{T} \subseteq \mathcal{C}$ is another Grothendieck category, the codomain of the **localization** of $\mathcal{C}$ at $\mathcal{T}$. This localization is an exact functor $(-)_{\mathcal{T}} : \mathcal{C} \to \mathcal{C}/\mathcal{T}$ to the Grothendieck category $\mathcal{C}/\mathcal{T}$ satisfying certain universal properties (cf. [16, Chapter 3]). A section functor $\sigma_{\mathcal{T}} : \mathcal{C}/\mathcal{T} \subseteq \mathcal{C}$ of the localization functor allows one to identify the category $\mathcal{C}/\mathcal{T}$ with the subcategory of $\mathcal{C}$ consisting of $\mathcal{T}$-torsion-free and $\mathcal{T}$-closed objects; recall that an object $M \in \mathcal{C}$ is $\boldsymbol{\mathcal{T}}$**-closed** if for every object $T \in \mathcal{T}$, $\mathrm{Ext}^1(T, M) = 0$. Because the section functor $\sigma_{\mathcal{T}}$ is the right adjoint of the exact functor $(-)_{\mathcal{T}}$, it preserves injective objects. Furthermore, the category $\mathcal{C}/\mathcal{T}$ viewed as a subcategory of $\mathcal{C}$ is closed under direct summands, so the section functor yields a continuous function that preserves [33, Proposition IV.9.5] indecomposable injective objects and thus induces a continuous function

$$\mathrm{Sp}(\sigma_{\mathcal{T}}) : \mathrm{Sp}(\mathcal{C}/\mathcal{T}) \to \mathrm{Sp}(\mathcal{C})$$

that consitutes a homeomorphism from $\mathrm{Sp}(\mathcal{C}/\mathcal{T})$ to its image $\mathcal{O}(\mathcal{T})^c$, the complement of the open subset associated to $\mathcal{T}$. We describe this state of affairs by expressing the Gabriel spectrum $\mathrm{Sp}(\mathcal{C})$ as a disjoint union

$$\mathrm{Sp}(\mathcal{C}) = \mathrm{Sp}(\mathcal{T}) \dot{\cup} \mathrm{Sp}(\mathcal{C}/\mathcal{T})$$

of Gabriel spectra, homeomorphic to an open and closed subset of $\mathrm{Sp}(\mathcal{C})$, respectively.

For example, the subcategory $\mathcal{T} = ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ is a hereditary torsion class of the Grothendieck category $\mathcal{C} = ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. The localization of $\mathcal{C}$ at $\mathcal{T}$ is given by the evaluation functor

$$\mathrm{Ev} : ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab}) \to R\text{-Mod},$$

defined by $F \mapsto F(R)$. We noted earlier how the hereditary torsion theory with torsion class $\mathcal{T}$ is cogenerated by the injective objects of the form $(-, E)$, as E ranges over the injective R-modules. Using the fact that this class of injective objects is closed under products, the $\mathcal{T}$-torsion-free, $\mathcal{T}$-closed objects of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ may be classified as those objects G that possess an injective resolution of the form

$$0 \longrightarrow G \longrightarrow (-, E_0) \xrightarrow{\eta} (-, E_1),$$

where E_0 and E_1 are injective R-modules. There is a morphism $f : E_0 \to E_1$ such that $\eta = (-, f)$, so if we let $M = \mathrm{Ker}\, f$, then $G = (-, M)$. This shows that the subcategory $\mathcal{C}/\mathcal{T}$ of $\mathcal{C}$ is nothing more than the category $\mathrm{Flat}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of flat functors, and that the section functor $\sigma_{\mathcal{T}}$ is equivalent to the functor $\Upsilon : R\text{-Mod} \to ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ given by $M \mapsto (-, M)$. The induced continuous function of Gabriel spectra

$$\mathrm{Sp}(\Upsilon) : \mathrm{Sp}(R\text{-Mod}) \to \mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$$

is given by $E \mapsto (-, E)$.

Similarly, the subcategory $\mathcal{T} = (\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ is a hereditary torsion class of Grothendieck category $\mathcal{C} = (\mathrm{mod}\text{-}R, \mathrm{Ab})$. The localization of $\mathcal{C}$ at $\mathcal{T}$ is also given by the evaluation functor $\mathrm{Ev} : (\mathrm{mod}\text{-}R, \mathrm{Ab}) \to R\text{-Mod}$. We noted earlier that this hereditary torsion theory with torsion class $\mathcal{T}$ is cogenerated by the injective objects of the form $-\otimes E$, as E ranges over the injective R-modules. Using the fact that this class of injective objects of (mod-R, Ab) is closed under products, the $\mathcal{T}$-torsion-free, $\mathcal{T}$-closed objects of (mod-R, Ab) may be classified as those objects G that possess an injective resolution of the form

$$0 \longrightarrow G \longrightarrow -\otimes E_0 \xrightarrow{\eta} -\otimes E_1,$$

where E_0 and E_1 are injective R-modules. There is a morphism $f : E_0 \to E_1$ such that $\eta = -\otimes f$, so if we let $M = \operatorname{Ker} f$, then the section functor, which we will denote by σ_R in this case, is given by $M \mapsto G$. The point is that the induced continuous function of Gabriel spectra

$$\mathrm{Sp}(\sigma_R) : \mathrm{Sp}(R\text{-Mod}) \to \mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$$

is given by $E \mapsto -\otimes E$.

These two situations are brought together by the following commutative diagram of continuous functions,

$$\begin{array}{ccc} \mathrm{Sp}(R\text{-Mod}) & \xrightarrow{\mathrm{Sp}(\Upsilon)} & \mathrm{Sp}(\text{mod-}R, \mathrm{Ab}) \\ \Vert & & \downarrow{\scriptstyle \Xi_R} \\ \mathrm{Sp}(R\text{-Mod}) & \xrightarrow{\mathrm{Sp}(\sigma_R)} & \mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab}). \end{array}$$

Let us summarize these findings as follows.

THEOREM 36: *There are partitions of the covariant and contravariant Gabriel spectra of R,*

$$\mathrm{Sp}(\text{mod-}R, \mathrm{Ab}) = \mathrm{Sp}(\underline{\text{mod}}\text{-}R, \mathrm{Ab}) \,\dot\cup\, \mathrm{Sp}(R\text{-Mod}) \text{ and}$$

$$\mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Sp}((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab}) \,\dot\cup\, \mathrm{Sp}(R\text{-Mod}),$$

each into an open subset and a closed subset. The canonical bijective correspondence $\Xi_R : \mathrm{Sp}(\text{mod-}R, \mathrm{Ab}) \to \mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ respects the partitions and induces a homeomorphism of the open subsets and closed subsets, respectively.

The contravariant functor category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and the covariant functor category (mod-R, Ab) give a natural example of a pair of Grothendieck categories with equivalent hereditary torsion classes $((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab})$ and ($\underline{\text{mod}}$-R, Ab) whose localizations are also equivalent. The ambient categories

$((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and $(\text{mod-}R, \mathrm{Ab})$ are themselves rarely equivalent. Indeed, an equivalence of these two categories exists if and only if there is an equivalence between $R\text{-mod} \cong \mathrm{proj}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ and $(\text{mod-}R)^{\mathrm{op}} \cong \mathrm{proj}(\text{mod-}R, \mathrm{Ab})$, that is, if and only if there a duality between R-mod and mod-R.

4. Purity in $R\text{-}\overline{\mathrm{Mod}}$

If the ring R is Quasi-Frobenius, then the category $R\text{-}\overline{\mathrm{Mod}}$ may be equipped with the structure of a triangulated category [11, 35, 36]. A theory of purity has been developed for triangulated categories by Beligiannis [8, 9] and H. Krause [25]. In this section, we use the results proved so far to show that if $m : M \to \mathrm{PE}(M)$ is the pure-injective envelope of M, then the induced morphism $\overline{m} : \overline{M} \to \overline{\mathrm{PE}(M)}$ on the stable classes is the $R\text{-}\overline{\mathrm{Pinj}}$-envelope of $\overline{M}$. In case R is Quasi-Frobenius, the objects of $R\text{-}\overline{\mathrm{Pinj}}$ are the pure-injective objects of $R\text{-}\overline{\mathrm{Mod}}$, considered as a triangulated category.

4.1. Quasi-Frobenius rings. The ring R is called **Quasi-Frobenius** (QF) [29] if it is left and right artinian and left and right self-injective. QF rings are really nice. If R is QF, then it is left perfect, so that every flat left R-module is projective. The category $R\text{-}\underline{\mathrm{Mod}}$, which we have defined in general to be the category R-Mod modulo morphisms that factor through a flat module, coincides with the usual definition of $R\text{-}\underline{\mathrm{Mod}}$ as the category of left R-modules modulo morphisms that factor through a projective. If R is QF, then it is left noetherian, so that every fp-injective left R-module is injective. Thus the category $R\text{-}\overline{\mathrm{Mod}}$, which we have defined to be the category R-Mod modulo morphisms that factor through an fp-injective module, coincides with the usual definition of $R\text{-}\overline{\mathrm{Mod}}$ as the category of left R-modules modulo morphisms that factor through an injective module. If M is an arbitrary R-module, we may decompose it as a direct sum

$$M = M_{\mathrm{if}} \oplus E_M,$$

where M_{if} is injective-free and E_M is injective. This is done by applying Zorn's Lemma to find a maximal fp-injective submodule of M, which is necessarily injective and yields E_M. Furthermore, if R is a QF ring, then a left R-module is projective if and only if it is injective. Thus

$$R\text{-}\underline{\mathrm{Mod}} = R\text{-}\overline{\mathrm{Mod}},$$

and the decomposition above has the property that E_M is projective and that M_{if} is projective-free.

If R is QF, then every module M has a projective cover $c : P \to M$, whose kernel is denoted $\Omega(M)$. The assignment $M \mapsto \Omega(M)$ induces a functor $\Omega : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ which is a self-equivalence of R-$\underline{\mathrm{Mod}}$ [11, Theorem 11.4]. The equivalence inverse $\Omega^{-1} : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is induced by the rule that associates to M the cokernel of the injective envelope $e : M \to E$. If M is a finitely presented module, then $\Omega(M)$ and $\Omega^{-1}(M)$ are also finitely presented, so the restriction of Ω to R-$\underline{\mathrm{mod}}$ is also a self-equivalence with equivalence inverse given by the restriction of Ω^{-1} to R-$\underline{\mathrm{mod}}$. These self-equivalences of R-$\underline{\mathrm{mod}}$ induce self-equivalences of the functor category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$, defined by $\Omega_*(F) = F(\Omega(-))$ and $\Omega_*^{-1}(F) = F(\Omega^{-1}(-))$. Given modules M and N, there is an isomorphism,

$$\mathrm{Ext}^1(M, N) \cong \underline{\mathrm{Hom}}(\Omega(\underline{M}), \underline{N}), \tag{5}$$

natural in both variables (cf. [11, Theorem 5.1]). Restricting the equivalence to finitely presented modules M implies that the functors $\mathrm{Ext}^1(-, N)$ and $\Omega_*[\underline{\mathrm{Hom}}(-, \underline{N})]$ are isomorphic as objects of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

When R is a QF ring, then $R\text{-}\underline{\mathrm{Mod}} = R\text{-}\overline{\mathrm{Mod}}$ has the structure of a triangulated category. A morphism $\underline{f} : \underline{M} \to \underline{N}$ is said to be a **pure-monomorphism** in $R\text{-}\underline{\mathrm{Mod}} = R\text{-}\overline{\mathrm{Mod}}$ provided that the morphism $\underline{\mathrm{Hom}}(-, \underline{f})$ is a monomorphism in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. This is equivalent to the definition of Belligianis [9, §4] and H. Krause [25, §1] using [22, Lemma 9.6.9]. The naturality of the isomorphisms (5) yields the equation $\mathrm{Ext}^1(-, f) = \Omega_*[\underline{\mathrm{Hom}}(-, \underline{f})]$, which implies that $\underline{f}$ is a pure-monomorphism if and only if $\mathrm{Ext}^1(-, f)$ is a monomorphism in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

4.2. Ext-MONOMORPHISMS. Bearing in mind the comments made above regarding QF rings, let us return to the general setting of an associative ring R. A morphism $\overline{f} : \overline{X} \to \overline{Y}$ in R-$\overline{\mathrm{Mod}}$ will be called an Ext-**monomorphism** if the morphism $\mathrm{Ext}^1(-, f) : \mathrm{Ext}^1(-, X) \to \mathrm{Ext}^1(-, Y)$ is a monomorphism in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. By Proposition 15, a monomorphism $f : X \to Y$ induces an Ext-monomorphism $\overline{f} : \overline{X} \to \overline{Y}$ if and only if it is a pure-monomorphism. Evidently, the composition of two Ext-monomorphisms is again such, and if $\overline{f} = \overline{g}\overline{h}$ is an Ext-monomorphism, then so is $\overline{h}$. Similarly, a morphism $\overline{f} : \overline{X} \to \overline{Y}$ will be called an Ext-**phantom** if $\mathrm{Ext}^1(-,f)=0$. The two concepts are related as follows.

PROPOSITION 37: *The functor*

$$R\text{-}\overline{\mathrm{Mod}} \to ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}),$$

given by $M \mapsto \mathrm{Ext}^1(-,M)$ *is faithful, i.e., there are no* Ext-*phantoms, if and only if every* Ext-*monomorphism is a monomorphism in* $R\text{-}\overline{\mathrm{Mod}}$*.*

Proof. Let $f: M \to N$ be a morphism in R-Mod. The **fp-injective preenvelope** $a: M \to A$ of M exists by [15, Propostion 6.2.4]. Consider the pushout diagram

$$\begin{array}{ccc} M & \xrightarrow{\ a\ } & A \\ {\scriptstyle f}\downarrow & & \vdots{\scriptstyle q} \\ N & \overset{p}{\cdots\cdots\to} & K. \end{array}$$

It gives rise to a short exact sequence

$$0 \longrightarrow M \xrightarrow{(f,-a)} N \oplus A \xrightarrow{\binom{p}{q}} K \longrightarrow 0,$$

which induces a long exact sequence of functors. Part of that long exact sequence is

$$\mathrm{Ext}^1(-,M) \xrightarrow{\mathrm{Ext}^1(-,f)} \mathrm{Ext}^1(-,N) \xrightarrow{\mathrm{Ext}^1(-,p)} \mathrm{Ext}^1(-,K).$$

If $\overline{f}$ is an Ext-phantom, then $\overline{p}$ is an Ext-monomorphism. Furthermore, the composition pf factors through the fp-injective module A, so that $\overline{p}\,\overline{f}=0$ in $R\text{-}\overline{\mathrm{Mod}}$. If every Ext-monomorphism is a monomorphism, then $\overline{f}=0$.

For the converse, suppose that every Ext-phantom is 0, and let $\overline{f}: \overline{N} \to \overline{K}$ be an Ext-monomorphism. If $\overline{g}: \overline{M} \to \overline{N}$ is such that $\overline{f}\,\overline{g}=0$, then $\mathrm{Ext}^1(-,f)\mathrm{Ext}^1(-,g)=0$. Since $\overline{f}$ is an Ext-monomorphism, $\overline{g}$ is an Ext-phantom, and so $\overline{g}=0$. ∎

Suppose that $\overline{f}: \overline{M} \to \overline{N}$ is a morphism in $R\text{-}\overline{\mathrm{Mod}}$ and let $\overline{n}: \overline{N} \to \overline{\mathrm{PE}(N)}$ be the morphism induced by the pure-injective envelope of N. Because $\overline{n}$ is an Ext-monomorphism, $\overline{f}$ is an Ext-phantom if and only if $\overline{n}\,\overline{f}$ is an Ext-phantom.

But by Theorem 16, there are no nonzero Ext-phantoms whose codomain is the class of a pure-injective module. Thus $\overline{f}$ is an Ext-phantom if and only if $\overline{n}\overline{f} = 0$.

Define an object $\overline{M} \in R\text{-}\overline{\mathrm{Mod}}$ to be Ext-**injective** if for every Ext-monomorphism $\overline{f} : \overline{X} \to \overline{Y}$ and morphism $\overline{g} : \overline{X} \to \overline{M}$, there is a morphism $\overline{h} : \overline{Y} \to \overline{M}$ such that the diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\ \overline{f}\ } & \overline{Y} \\ {\scriptstyle \overline{g}}\big\downarrow & \swarrow {\scriptstyle \overline{h}} & \\ \overline{M} & & \end{array}$$

commutes. If R is a QF ring, and $R\text{-}\overline{\mathrm{Mod}}$ is regarded as a triangulated category, then the Ext-injective objects are precisely the pure-injective objects of $R\text{-}\overline{\mathrm{Mod}}$ [25, Proposition 1.15]. If M is a pure-injective R-module and $\overline{f} : \overline{X} \to \overline{Y}$ and $\overline{g} : \overline{X} \to \overline{M}$ are given as in the definition, then, because $\mathrm{Ext}^1(-, M)$ is an injective object of $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$, there is a morphism $\chi : \mathrm{Ext}^1(-, Y) \to \mathrm{Ext}^1(-, M)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(-,X) & \xrightarrow{\ \mathrm{Ext}^1(-,f)\ } & \mathrm{Ext}^1(-,Y) \\ {\scriptstyle \mathrm{Ext}^1(-,g)}\big\downarrow & \swarrow {\scriptstyle \chi} & \\ \mathrm{Ext}^1(-,M) & & \end{array}$$

commutes. By Theorem 16, there is a morphism $h : Y \to M$ such that $\chi = \mathrm{Ext}^1(-, h)$. Similarly, Theorem 16 implies that $\overline{hf} = \overline{g}$. It follows that every object $\overline{M} \in R\text{-}\overline{\mathrm{Pinj}}$ is Ext-injective.

Proposition 38: *An object $\overline{M}$ of $R\text{-}\overline{\mathrm{Mod}}$ is* Ext-*injective if and only if it belongs to $R\text{-}\overline{\mathrm{Pinj}}$.*

Proof. One direction has just been proved, so assume that $\overline{M}$ is Ext-injective and consider the morphism $\overline{m} : \overline{M} \to \overline{\mathrm{PE}(M)}$ in $R\text{-}\overline{\mathrm{Mod}}$ induced by the pure-injective envelope of M in R-Mod. There is a morphism $\overline{g} : \overline{\mathrm{PE}(M)} \to \overline{M}$ that makes the diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{m}} & \overline{\mathrm{PE}(M)} \\ \| & \swarrow_{\overline{g}} & \\ \overline{M} & & \end{array}$$

commute. Let us show that $\overline{m}$ is an isomorphism with inverse $\overline{g}$. By the commutativity of the diagram, $\overline{g}$ is the left inverse of $\overline{m}$. Consider the corresponding commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(-,M) & \xrightarrow{\mathrm{Ext}^1(-,m)} & \mathrm{Ext}^1(-,\mathrm{PE}(M)) \\ \| & \swarrow_{\mathrm{Ext}^1(-,g)} & \\ \mathrm{Ext}^1(-,M) & & \end{array}$$

in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. By the commutativity of the diagram, $\mathrm{Ext}^1(-,g)$ is an epimorphism. Since $\mathrm{Ext}^1(-,m) : \mathrm{Ext}^1(-,M) \to \mathrm{Ext}^1(-,\mathrm{PE}(M))$ is the injective envelope of $\mathrm{Ext}^1(-,M)$, and the restriction of $\mathrm{Ext}^1(-,g)$ to $\mathrm{Ext}^1(-,M)$ is the identity, the morphism $\mathrm{Ext}^1(-,g)$ is also a monomorphism. Thus $\mathrm{Ext}^1(-,g)$ and consequently $\mathrm{Ext}^1(-,m)$ are both isomorphism. Because $\mathrm{PE}(M)$ is pure-injective, Theorem 16 implies that $\overline{mg} = 1_{\overline{M}}$. ∎

The proof of the proposition also shows that if $\overline{M} \in R\text{-}\overline{\mathrm{Pinj}}$ and $\overline{f} : \overline{M} \to \overline{N}$ is an Ext-monomorphism, then it is a split monomorphism.

Theorem 39: *If $\overline{M}$ belongs to R-$\overline{\mathrm{Mod}}$, then the R-$\overline{\mathrm{Pinj}}$-envelope of $\overline{M}$ is given by the morphism $\overline{m} : \overline{M} \to \overline{\mathrm{PE}(M)}$ induced by the pure-injective envelope of M. Regarded as a morphism, the R-$\overline{\mathrm{Pinj}}$-envelope $\overline{M}$ is an* Ext*-monomorphism.*

Proof. The second statement follows from Corollary 31. Let $\overline{f} : \overline{M} \to \overline{N}$ be a morphism in R-$\overline{\mathrm{Mod}}$ with $\overline{N}$ in R-$\overline{\mathrm{Pinj}}$. Because we may take N pure-injective, there is a morphism $g : \mathrm{PE}(M) \to N$ such that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(-,M) & \xrightarrow{\mathrm{Ext}^1(-,m)} & \mathrm{Ext}^1(-,\mathrm{PE}(M)) \\ {\scriptstyle \mathrm{Ext}^1(-,f)}\downarrow & \swarrow_{\mathrm{Ext}^1(-,g)} & \\ \mathrm{Ext}^1(-,N) & & \end{array}$$

in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ commutes. By Theorem 16, $\overline{f} = \overline{gm}$, and so $\overline{f}$ factors through $\overline{m}$. This shows that $\overline{m} : \overline{M} \to \overline{\mathrm{PE}(M)}$ is an R-$\overline{\mathrm{Pinj}}$-preenvelope of $\overline{M}$.

To verify that $\overline{m} : \overline{M} \to \overline{\mathrm{PE}(M)}$ is the R-$\overline{\mathrm{Pinj}}$-envelope of $\overline{M}$, suppose that $\overline{f} : \overline{\mathrm{PE}(M)} \to \overline{\mathrm{PE}(M)}$ is an endomorphism of $\overline{\mathrm{PE}(M)}$ over $\overline{M}$. Then the diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(-,M) & \xrightarrow{\mathrm{Ext}^1(-,m)} & \mathrm{Ext}^1(-,\mathrm{PE}(M)) \\ {\scriptstyle \mathrm{Ext}^1(-,m)}\downarrow & \swarrow {\scriptstyle \mathrm{Ext}^1(-,f)} & \\ \mathrm{Ext}^1(-,\mathrm{PE}(M)) & & \end{array}$$

in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ commutes. Since $\mathrm{Ext}^1(-,\mathrm{PE}(M))$ is the injective envelope of $\mathrm{Ext}^1(-,M)$, the morphism $\mathrm{Ext}^1(-,f)$ is an isomorphism. By Theorem 16, the morphism $\overline{f}$ is an isomorphism in $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$. ∎

5. Hereditary torsion classes of finite type

Notation: Throughout this section, $\mathcal{C}$ will denote a locally finitely generated Grothendieck category.

The advantage of this hypothesis is that the category $\mathcal{C}$ is cogenerated by the indecomposable injective objects. In particular, the open subset $\mathcal{O}(\mathcal{T})$ associated to a nonzero hereditary torsion class $\mathcal{T} \subseteq \mathcal{C}$ is nonempty.

An object $C \in \mathcal{C}$ is **finitely presented** if it is finitely generated and every epimorphism $g : B \to C$ with B finitely generated has a finitely generated kernel. The subcategory of finitely presented objects of $\mathcal{C}$ is denoted by $\mathrm{fp}(\mathcal{C})$. A hereditary torsion class $\mathcal{T} \subseteq \mathcal{C}$ is **of finite type** if it is generated by a set of finitely presented objects of $\mathcal{C}$,

$$\mathcal{T} = \mathrm{Gen}(\mathcal{T} \cap \mathrm{fp}(\mathcal{C})).$$

Proposition 40: *If $\mathcal{T} \subseteq \mathcal{C}$ is a hereditary torsion theory of finite type, and $F \in \mathcal{C}$ is finitely generated, then the localization $F_{\mathcal{T}}$ is finitely generated in $\mathcal{C}/\mathcal{T}$. The localization $\mathcal{C}/\mathcal{T}$ is locally finitely generated.*

Proof. The second statement follows from the first together with the fact that the localization functor preserves direct limits. If $C \in \mathcal{C}$ and $\mathcal{T}(C)$ denotes the $\mathcal{T}$-subobject of C, then there is a short exact sequence [16, Chapter III.1]

$$0 \longrightarrow C' \longrightarrow C_{\mathcal{T}} \longrightarrow G \longrightarrow 0,$$

where $C' = C/\mathcal{T}(C)$ and $G \in \mathcal{T}$. If C is finitely generated, then so is C'. To prove that $C_{\mathcal{T}}$ is finitely generated in $\mathcal{C}/\mathcal{T}$, it must be shown that whenever a directed union

$$\sum_{i \in I} F_i \subseteq C_{\mathcal{T}}$$

of $\mathcal{T}$-torsion-free, $\mathcal{T}$-closed subobjects of $C_{\mathcal{T}}$ has the property that the quotient $C_{\mathcal{T}}/\Sigma_i F_i$ belongs to $\mathcal{T}$, then there is a $k \in I$ such that $C_{\mathcal{T}}/F_k \in \mathcal{T}$.

Since $C_{\mathcal{T}}/C' \in \mathcal{T}$, it follows that $(C' + \Sigma_i F_i)/\Sigma_i F_i$ is an object of $\mathcal{T}$, which is finitely generated, because it is a quotient of C'. Now

$$(C' + \Sigma_i F_i)/\Sigma_i F_i = C'/(C' \cap \Sigma_i F_i) = C'/\Sigma_i(C' \cap F_i).$$

As $\mathcal{T}$ is of finite type, there is an object $T \in \mathcal{T}$, finitely presented in $\mathcal{C}$, and an epimorphism $f : T \mapsto C'/\Sigma_i(C' \cap F_i)$. As T is finitely presented, there is a $j \in I$ such that the morphism f factors according to the commutative diagram

$$\begin{array}{ccc} & & C'/(C' \cap F_j) \\ & \overset{g}{\nearrow} & \Big\downarrow p \\ T & \xrightarrow{f} & C'/\Sigma_i(C' \cap F_i). \end{array}$$

Working in $C'/(C' \cap F_j)$, we have the equality $[\operatorname{Im} g + \Sigma_{i \geq j} (C' \cap F_i)]/(C' \cap F_j) = C'/(C' \cap F_j)$. Because this object is finitely generated, there is a $k \geq j$ such that

$$[\operatorname{Im} g + (C' \cap F_k)]/(C' \cap F_j) = C'/(C' \cap F_j).$$

As $\operatorname{Im} g$ belongs to $\mathcal{T}$, so does $C'/(C' \cap F_k)$. But $C_{\mathcal{T}}/C'$ is also an object of $\mathcal{T}$, so that $C_{\mathcal{T}}/(C' \cap F_k) \in \mathcal{T}$ and hence $C_{\mathcal{T}}/F_k \in \mathcal{T}$. ∎

One consequence of Proposition 40 is that the Gabriel spectrum of $\mathcal{C}$ detects proper inclusion of finite type hereditary torsion classes.

PROPOSITION 41: *Let $\mathcal{T} \subseteq \mathcal{T}'$ be a proper inclusion of hereditary torsion classes of $\mathcal{C}$ with $\mathcal{T}$ of finite type, then the inclusion $\mathcal{O}(\mathcal{T}) \subseteq \mathcal{O}(\mathcal{T}')$ of open subsets in the Gabriel spectrum $\mathrm{Sp}(\mathcal{C})$ is proper.*

Proof. Let $C \in \mathcal{T}'$ be an object that does not belong to $\mathcal{T}$. Then there is a finitely generated subobject $A \subseteq C$ that does not belong to $\mathcal{T}$. By the proof of Proposition 40, the localization $A_{\mathcal{T}}$ is a nonzero finitely generated object of the locally finitely generated Grothendeick category $\mathcal{C}/\mathcal{T}$. The finitely generated object $A_{\mathcal{T}}$ has a simple quotient object $X \in \mathcal{C}/\mathcal{T}$. Then $E(X)$ is a point in $\mathcal{O}(\mathcal{T}')$ that does not belong to $\mathcal{O}(\mathcal{T})$. ∎

Suppose that R is a simple von Neumann regular ring that is not artinian. Then the hereditary torsion class $\mathcal{T} \subseteq R$-Mod of semi-artinian modules is proper, but it is not known whether the open subset $\mathcal{O}(\mathcal{T})$ is always proper in Sp(R-Mod) (cf. [34]). By Proposition 41, this is only interesting when $\mathcal{T}$ is not of finite type.

Proposition 42: *Let $\mathcal{T} \subseteq \mathcal{C}$ be a hereditary torsion theory of finite type. If $M \in \mathcal{C}$ is fp-injective, then the $\mathcal{T}$-torsion subobject of M is fp-injective in $\mathcal{T}$.*

Proof. Denote by $\mathcal{T}(M)$ the $\mathcal{T}$-torsion subobject of M and let $C \in \mathcal{T}$ be finitely presented. Consider an extension of $\mathcal{T}(M)$ by C and take the pushout with the embedding $\mathcal{T}(M) \subseteq M$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}(M) & \longrightarrow & N & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & \overset{g}{\swarrow} & & & & & \\ & & M. & & & & & & \end{array}$$

Because $\mathcal{T}$ is of finite type, the object C is finitely presented in $\mathcal{C}$, and M is fp-injective, the embedding of $\mathcal{T}(M)$ into M extends to the morphism $g : N \to M$ as indicated by the dotted arrow. Now $N \in \mathcal{T}$, so that the image of g must lie in $\mathcal{T}(M)$. ∎

5.1. The Ziegler topology. An object of $\mathcal{C}$ is **coherent** if it is finitely presented and every finitely generated subobject is also finitely presented. The category $\mathcal{C}$ is said to be **locally coherent** [20, 26] if there is a generating set of coherent objects. In that case, every finitely presented object is coherent and the category fp($\mathcal{C}$) is abelian. Conversely, if $\mathcal{C}$ is locally finitely presented and fp($\mathcal{C}$) is abelian, then every finitely presented object is coherent. A ring R is **left coherent** if the category R-Mod is a locally coherent Grothendieck category. One advantage of the category (mod-R, Ab) of covariant functors on mod-R is that it is a locally coherent Grothendieck category for any associative ring R.

If the Grothendieck category $\mathcal{C}$ is locally coherent, then the subsets $\mathcal{O}(\mathcal{T})$, as $\mathcal{T} \subseteq \mathcal{C}$ ranges over the hereditary torsion classes of finite type, satisfy [20, Theorem 3.4] the axioms for an algebra of open subsets of a topology on $\mathrm{Sp}(\mathcal{C})$. The set of indecomposable injective objects of $\mathcal{C}$ endowed with this topology is known as the **Ziegler spectrum of** $\mathcal{C}$, and is denoted by $\mathrm{Zg}(\mathcal{C})$. The open subsets of $\mathrm{Zg}(\mathcal{C})$ are in bijective correspondence $\mathcal{T} \mapsto \mathcal{O}(\mathcal{T})$ (cf. [20, Theorem 2.8]) with the hereditary torsion classes $\mathcal{T} \subseteq \mathcal{C}$ of finite type.

If $\mathcal{C}$ is locally coherent and $\mathcal{T} \subseteq \mathcal{C}$ is of finite type, then the Grothendieck categories $\mathcal{T}$ and $\mathcal{C}/\mathcal{T}$ are also locally coherent [20, §3]. They yield a partition of the Ziegler spectrum of $\mathcal{C}$ into the open subset $\mathcal{O}(\mathcal{T})$ and its complement, which are canonically homeomorphic to the Ziegler spectrum of $\mathcal{T}$ and that of the localization $\mathcal{C}/\mathcal{T}$, respectively,

$$\mathrm{Zg}(\mathcal{C}) = \mathrm{Zg}(\mathcal{T}) \dot{\cup} \mathrm{Zg}(\mathcal{C}/\mathcal{T}).$$

PROPOSITION 43: [2, §III.2] *The following are equivalent for a ring R :*

(1) *the ring R is left coherent;*
(2) *the category* $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ *is locally coherent;*
(3) *the hereditary torsion class* $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab}) \subseteq (\mathrm{mod}\text{-}R, \mathrm{Ab})$ *is of finite type.*

Hereditary torsion theories of finite type play a prominent role in the theory of locally coherent Grothendieck categories. The argument just before Proposition 14 implies that the hereditary torsion class $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}) \subseteq ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is of finite type, whether or not the ring R is left coherent.

COROLLARY 44: *If R is a left coherent ring, then for every R-module, the object* $\mathrm{Ext}^1(-, M)$ *is fp-injective in* $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$.

Proof. Since R is left coherent, the hereditary torsion class $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab}) \subseteq (\mathrm{mod}\text{-}R, \mathrm{Ab})$ is of finite type. The functor $-\otimes M$ is fp-injective in $(\mathrm{mod}\text{-}R, \mathrm{Ab})$, so Theorem 42 implies that the torsion subobject $t(-\otimes M)$ is fp-injective in $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$. The equivalence $\mathrm{Tr}_* : (\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab}) \to ((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ associates to $t(-\otimes M)$ the contravariant functor $\mathrm{Ext}^1(-, M)$, By Theorem 29. ∎

Given any associative ring R, there is a covariant Ziegler spectrum $\mathrm{Zg}(\mathrm{mod}\text{-}R, \mathrm{Ab})$, because the category $(\mathrm{mod}\text{-}R, \mathrm{Ab})$ is locally coherent. The Ziegler spectrum $\mathrm{Zg}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of the category of contravariant functors on R-mod only makes sense if the ring R is left coherent. In that case, there is a partition of the category of covariant functors

$$\mathrm{Zg}(\mathrm{mod}\text{-}R, \mathrm{Ab}) = \mathrm{Zg}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab}) \mathbin{\dot\cup} \mathrm{Zg}(R\text{-}\mathrm{Mod})$$

as a disjoint union of a Ziegler open subset, because $(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$ is of finite type, and a Ziegler closed subset. Similarly, there is a partition of the category of contravariant functors

$$\mathrm{Zg}((R\text{-}\mathrm{mod})^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Zg}((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab}) \mathbin{\dot\cup} \mathrm{Zg}(R\text{-}\mathrm{Mod}).$$

The canonical bijective correspondence Ξ_R of Gabriel spectra given by (4) respects these partitions and induces by restriction a homeomorphism of the respective open subsets, as well as a homeomorphism of the closed subsets.

5.2. Serre subcategories of $\mathrm{fp}(\mathcal{C})$. If $\mathcal{C}$ is locally coherent and $\mathcal{T} \subseteq \mathcal{C}$ is a hereditary torsion class, then the intersection $\mathcal{S}(\mathcal{T}) = \mathcal{T} \cap \mathrm{fp}(\mathcal{C})$ is a **Serre subcategory** of $\mathrm{fp}(\mathcal{C})$ in the sense that whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence in $\mathrm{fp}(\mathcal{C})$, then $B \in \mathcal{S}$ if and only if A and C belong to $\mathcal{S}$. If $\mathcal{T}$ is of finite type, then $\mathcal{S}(\mathcal{T})$ provides a set of generators of $\mathcal{T}$. In the other direction we have that if $\mathcal{S} \subseteq \mathrm{fp}(\mathcal{C})$ is a Serre subcategory, then the torsion theory $\mathrm{Gen}(\mathcal{S})$ of objects generated by $\mathcal{S}$ is hereditary [20, Theorem 2.5]. Thus the Ziegler interior of a Gabriel open subset $\mathcal{O}(\mathcal{T})$ is the open subset associated to the hereditary torsion class $\mathrm{Gen}(\mathcal{S}(\mathcal{T}))$, and $\mathcal{T}$ is of finite type if and only if

$$\mathcal{T} = \mathrm{Gen}(\mathcal{S}(\mathcal{T})).$$

If $M \in \mathcal{C}$ is an fp-injective object, then applying the functor $(-, M)$ to the short exact sequence in $\mathrm{fp}(\mathcal{C})$ above yields a short exact sequence

$$0 \longrightarrow (C, M) \longrightarrow (B, M) \longrightarrow (A, M) \longrightarrow 0$$

of abelian groups. Thus the subcategory $\mathcal{S}(M) \subseteq \mathrm{fp}(\mathcal{C})$ defined by

$$\mathcal{S}(M) := \{A \in \mathrm{fp}(\mathcal{C}) : (A, M) = 0\}$$

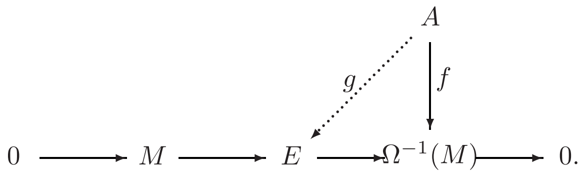
is a Serre subcategory of $\mathrm{fp}(\mathcal{C})$. If $\mathcal{S} \subseteq \mathcal{S}(M)$, then M is $\mathrm{Gen}(\mathcal{S})$-torsion-free and $\mathrm{Gen}(\mathcal{S})$-closed [20, Proposition 3.10], so that $M = M_{\mathrm{Gen}(\mathcal{S})}$ belongs to the localization $\mathcal{C}/\mathrm{Gen}(\mathcal{S})$. It is also fp-injective (ibid) as an object of the localization.

The next result, which does not require that $\mathcal{C}$ be locally coherent, will be useful in obtaining an upper bound on the injective dimension of an fp-injective object $M \in \mathcal{C}$ when $\mathcal{C}$ *is* locally coherent.

PROPOSITION 45: *If M is an fp-injective object of $\mathcal{C}$ and A is a finitely presented noetherian object of $\mathcal{C}$, then*

$$\mathrm{Hom}[A, \Omega^{-1}(M)] = 0.$$

Proof. Let $f : A \to \Omega^{-1}(M)$ be a morphism. Since M is fp-injective, this morphism lifts to the injective envelope E of M as indicated by the dotted arrow,



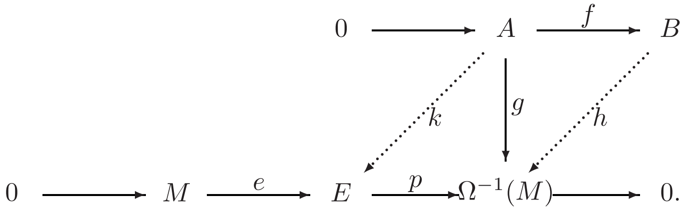
Every subobject of A is finitely generated, so every quotient of A is finitely presented. Thus the object K in the short exact sequence

$$0 \longrightarrow M \longrightarrow M + g(A) \longrightarrow K \longrightarrow 0$$

is finitely presented. Since M is fp-injective, the sequence must split. But $M \subseteq M + g(A) \subseteq E$ is an essential extension of M. It follows that $g(A) \subseteq M$, and hence that $f = 0$. ∎

PROPOSITION 46: *If $\mathcal{C}$ is locally coherent and $M \in \mathcal{C}$ an fp-injective object, then $\Omega^{-1}(M)$ is an fp-injective object of $\mathcal{C}$.*

Proof. To begin, let us show that if a monomorphism $f : A \to B$ is given, with A finitely presented, then any morphism $g : A \to \Omega^{-1}(M)$ extends to $h : B \to \Omega^{-1}(M)$ as indicated by the dotted arrow



The morphism $k : A \to E$ exists, because M is fp-injective. As E is injective, the morphism $k : A \to E$ extends to a morphism $q : B \to E$ (not shown) such that $k = qf$. Let $h = pq$. Then $g = pk = pqf = hf$.

Now suppose that there is a short exact sequence

$$0 \longrightarrow \Omega^{-1}(M) \longrightarrow X \xrightarrow{\ p\ } C \longrightarrow 0,$$

with C finitely presented. Because the category $\mathcal{C}$ is locally coherent, there is a morphism $g : B \to X$ from a coherent object B such that the composition $pg : B \to C$ is an epimorphism. As C is finitely presented, the kernel $A = \operatorname{Ker}(pg)$ is finitely generated. As B is coherent, A is finitely presented. One obtains a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ k\ } & B & \xrightarrow{\ pg\ } & C & \longrightarrow & 0, \\
 & & \downarrow & \swarrow & \downarrow g & \swarrow & \| & & \\
0 & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & X & \xrightarrow{\ p\ } & C & \longrightarrow & 0.
\end{array}$$

The dotted arrow in the left commutative square exists due to the argument from the previous paragraph. It induces the dotted arrow in the right commutative square, which yields a section of $p : X \to C$. ∎

Let us note that if $\mathcal{C}$ is locally coherent and $M \in \mathcal{C}$ is fp-injective, then

$$\mathcal{S}(M) \subseteq \mathcal{S}(\Omega^{-1}(M)).$$

If $A \in \mathcal{C}$ is finitely presented and $f : A \to \Omega^{-1}(M)$ is nonzero then, because M is fp-injective, this morphism will factor through $p : E \to \Omega^{-1}(M)$ as indicated by the dotted arrow,

$$\begin{array}{ccccccccc}
 & & & & & & A & & \\
 & & & & & \overset{g}{\swarrow} & \downarrow f & & \\
0 & \longrightarrow & M & \xrightarrow{\ e\ } & E & \xrightarrow{\ p\ } & \Omega^{-1}(M) & \longrightarrow & 0.
\end{array}$$

Since $g : A \to E$ is nonzero, we obtain $\mathcal{S}(E) \subseteq \mathcal{S}(\Omega^{-1}(M))$. But because $\operatorname{Gen}(\mathcal{S}(M))$ is hereditary of finite type and M is fp-injective, $\mathcal{S}(E) = \mathcal{S}(M)$.

Let $\mathcal{S} \subseteq \text{fp}(\text{mod-}R, \text{Ab})$ be a Serre subcategory. Define $\mathcal{S}' \supseteq \mathcal{S}$ to be the subcategory of $\text{fp}(\text{mod-}R, \text{Ab})$ that consists of the objects $A \in \text{fp}(\text{mod-}R, \text{Ab})$ for which the localized object $A_{\text{Gen}(\mathcal{S})}$ is noetherian in the localization of $(\text{mod-}R, \text{Ab})$ at $\text{Gen}(\mathcal{S})$. Because $A_{\text{Gen}(\mathcal{S})}$ is finitely presented in the localization $(\text{mod-}R, \text{Ab})/\text{Gen}(\mathcal{S})$, the subcategory $\mathcal{S}' \subseteq \text{fp}(\text{mod-}R, \text{Ab})$ is also a Serre subcategory. Define $\mathcal{S}^{(n)}$ for $n > 1$, by recursion as $\mathcal{S}^{(n+1)} := (\mathcal{S}^{(n)})'$. An R-module N has **finite elementary Krull dimension** [30, p. 217] if there is a natural number n such that $[\mathcal{S}(- \otimes_R N)]^{(n)} = \text{fp}(\text{mod-}R, \text{Ab})$. In that case, define the **elementary Krull dimension** of M, $\text{eK} - \dim(M)$, to be the least whole number n such that $[\mathcal{S}(- \otimes M)]^{(n+1)} = \text{fp}(\text{mod-}R, \text{Ab})$.

THEOREM 47: *Let $\mathcal{C}$ be locally coherent and $M \in \mathcal{C}$ an fp-injective object. Then*

$$[\mathcal{S}(M)]^{(n)} \subseteq \mathcal{S}[\Omega^{-n}(M)].$$

So if M has finite elementary Krull dimension, then $\text{eK} - \dim(M) \geq \dim(M)$, *the injective dimension of M.*

Proof. The inclusion is proved by induction on n. To prove the case $n = 1$, let $\mathcal{S} = \mathcal{S}(M)$ and suppose that $A \in \text{fp}(\mathcal{C})$ is such that $A_{\text{Gen}(\mathcal{S})}$ is noetherian in $\mathcal{C}/\text{Gen}(\mathcal{S})$. All the terms in the short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow \Omega^{-1}(M) \longrightarrow 0$$

are fp-injective. Furthermore, $\mathcal{S} = \mathcal{S}(E) \subseteq \mathcal{S}(\Omega^{-1}(M))$, so that this sequence may be thought of a short exact sequence of fp-injective objects in the localization $\mathcal{C}/\text{Gen}(\mathcal{S})$. We need to prove that $(A, \Omega^{-1}(M)) = 0$. Since $\Omega^{-1}(M)$ belongs to the localization $\mathcal{C}/\text{Gen}(\mathcal{S})$, the universal property of localization implies that $(A, \Omega^{-1}(M)) = (A_{\text{Gen}(\mathcal{S})}, \Omega^{-1}(M))$. Now $A_{\text{Gen}(\mathcal{S})}$ is a noetherian finitely presented object of the locally coherent Grothendieck category $\mathcal{C}/\text{Gen}(\mathcal{S})$, so that Proposition 46 applies to yield $(A_{\text{Gen}(\mathcal{S})}, \Omega^{-1}(M)) = 0$.

To prove the induction step, assume that $[\mathcal{S}(M)]^{(n)} \subseteq \mathcal{S}[\Omega^{-n}(M)]$. Then

$$[\mathcal{S}(M)]^{(n+1)} \subseteq (\mathcal{S}[\Omega^{-n}(M)])' \subseteq \mathcal{S}(\Omega^{-1}(\Omega^{-n}(M))) = \mathcal{S}[\Omega^{-(n+1)}(M)].$$

The first inclusion is obvious, while the second follows from the case $n = 1$ together with Proposition 46. If $\text{eK} - \dim(M) = n$, then

$$\mathcal{S}[\Omega^{-(n+1)}(M)] = [\mathcal{S}(M)]^{(n+1)} = \text{fp}(C).$$

It follows that $\Omega^{-(n+1)}(M) = 0$ and hence that $\dim(M) \leq n$. ∎

If $\mathcal{C} = (\text{mod-}R, \text{Ab})$ and N is an R-module, then the object $- \otimes N$ is fp-injective in $\mathcal{C}$ and its injective dimension is the pure-injective dimension of N (cf. [32]). The elementary Krull dimension of $- \times N$, if finite, is therefore a bound of the pure-injective dimension of N. Recall that the ring R is called **left pure-semisimple** if every left R-module is pure-injective. Equivalently, every fp-injective object $- \otimes M$ of $(\text{mod-}R, \text{Ab})$ is injective. If R is left coherent, then the analogous condition on the category $(\underline{\text{mod}}\text{-}R, \text{Ab})$ may be characterized as follows.

Theorem 48: *Consider the following conditions on a ring R:*

(1) *for every module M, the quotient $\text{PE}(M)/M$ is fp-injective;*
(2) *for every module M, the object $\text{Ext}^1(-, M)$ is injective in*

$$((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab});$$

(3) *the category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is locally noetherian.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If R is left coherent, the conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2). Consider the long exact sequence of functors induced by the short exact sequence

$$0 \longrightarrow M \longrightarrow \text{PE}(M) \longrightarrow Z \longrightarrow 0,$$

where the monomorphism is the pure-injective envelope of M and $Z = \text{PE}(M)/M$. The sequence is pure-exact, so the first connecting map is 0,

$$(-, Z) \xrightarrow{0} \text{Ext}^1(-, M) \longrightarrow \text{Ext}^1(-, \text{PE}(M)) \longrightarrow \text{Ext}^1(-, Z),$$

and because Z is fp-injective, $\text{Ext}^1(-, Z) = 0$. It follows that $\text{Ext}^1(-, M) \cong \text{Ext}^1(-, \text{PE}(M))$ is injective.

(2) $\Rightarrow$ (3). By [12, Propsition 1.1.2], $\coprod_i \text{Ext}^1(-, M_i) \cong \text{Ext}^1(-, \bigoplus_i M_i)$. The hypothesis thus implies that a coproduct of injective objects in $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is injective. By [33, Propostion V.4.3], the category $((R\text{-}\underline{\text{mod}})^{\text{op}}, \text{Ab})$ is locally noetherian.

(3) $\Rightarrow$ (1), assuming R is left coherent. The hypothesis is equivalent to the statement that $(\underline{\text{mod}}\text{-}R, \text{Ab})$ is locally noetherian. Since R is left coherent, the hereditary torsion class $(\underline{\text{mod}}\text{-}R, \text{Ab})$ is of finite type, and so every noetherian object in it is finitely presented in $(\text{mod-}R, \text{Ab})$. If M is an R-module, then

$-\otimes M$ is an fp-injective object of $(\text{mod-}R, \text{Ab})$ whose injective envelope is given by the monomorphism in the short exact sequence

$$0 \longrightarrow -\otimes M \longrightarrow -\otimes \mathrm{PE}(M) \longrightarrow -\otimes Z \longrightarrow 0,$$

where $Z = \mathrm{PE}(M)/M$. By Theorem 45, $\mathrm{Hom}[A, -\otimes Z] = 0$ for every $A \in (\underline{\text{mod}}\text{-}R, \text{Ab})$. By Theorem 29, $t(-\otimes Z) = \mathrm{Tr}_*[\mathrm{Ext}^1(-, Z)] = 0$, and so Z is fp-injective. ∎

6. Examples

In this section, some aspects of the covariant and contravariant Gabriel spectra of R are described for specific classes of rings. For example, it is shown that if R is left semihereditary, then the closed subset $\mathrm{Sp}(R\text{-Mod})$ is open in both the covariant as well as the contravariant Gabriel spectrum of R. Since the canonical bijective correspondence Ξ_R is a homeomorphism restricted to both $\mathrm{Sp}(R\text{-Mod})$ and its open complement, it is a homeomorphism of the two Gabriel spectra of R.

Proposition 49: *Let $I \subseteq R$ be a maximal left ideal and $E = E(R/I)$ the injective envelope of the simple R-module R/I. Then the indecomposable injective functor $(-, E)$ is the injective envelope in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of a simple object. The point $(-, E)$ is therefore isolated in the Gabriel spectrum of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. If $(-, E)$ is the injective envelope of a simple object X, then it is the only injective indecomposable that belongs to the open subset associated to the minimum hereditary torsion theory containing X; this is the class of objects all of whose quotients have a copy of X in the socle. To find such a simple functor, denote the inclusion morphism by $i : I \to R$ and let $X = \mathrm{Coker}\,(-, i)$. The exact sequence

$$0 \longrightarrow I \xrightarrow{\ i\ } R \longrightarrow E$$

in R-Mod yields an exact sequence

$$0 \longrightarrow (-, I) \xrightarrow{(-,i)} (-, R) \longrightarrow (-, E),$$

in $((R\text{-mod})^{\rm op}, \mathrm{Ab})$. The functor X is isomorphic to a subfunctor of $(-, E)$. Furthermore, it is obvious that $X(R) = R/I$ is simple as a left R-module. To see that X is simple, suppose that $Y \subseteq X$ is a proper nonzero subfunctor. The epimorphism $(-, R) \to X$ composed with the quotient map $X \to X/Y$ yields a nonzero morphism from $(-, R)$ to X/Y. By Yoneda's Lemma, $(X/Y)(R) = X(R)/Y(R) \neq 0$. Now $X(R) = R/I$ is a simple R-module, which forces $Y(R) = 0$. Since $Y \subseteq X$ is torsion-free, it leaves $Y = 0$ as the only possibility. ∎

If $E = E(S)$ is the injective envelope of a simple module S, then the point $- \otimes E$ of the covariant Gabriel spectrum $\mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$ is also isolated. Let $f : E \to E/S$ be the natural quotient map and denote by Y the kernel of the morphism $- \otimes f : - \otimes E \to - \otimes E/S$. It is a simple object of the category $(\text{mod-}R, \mathrm{Ab})$. For, suppose that $\mu : - \otimes E \to F$ were a morphism with nonzero kernel. Composing μ with the injective envelope $\eta : F \to - \otimes M$ yields a morphism $- \otimes g : - \otimes E \to - \otimes M$ which is not a split monomorphism, since its kernel is also nonzero. But then $g : E \to M$ is also not a split monomorphism and so its kernel must contain S. It follows that the kernel of $- \otimes g$ contains Y, which implies that Y must be simple.

Example 1: The ring R is **left semiartinian** if every nonzero R-module has nonzero socle. In particular, every indecomposable injective R-module is the envelope of a simple module. Then $\mathrm{Sp}(R\text{-Mod})$ is an open discrete subset of both the contravariant and covariant Gabriel spectra. The two spectra are therefore homeomorphic.

If R is left coherent and the maximal left ideal I is finitely generated, then the simple functor $\mathrm{Coker}\,(-, i)$ in the proof of Proposition 49 is finitely presented. The injective indecomposable $(-, E)$, where $E = E(R/I)$, is then also isolated in the contravariant Ziegler spectrum of R, as well as in the contravariant Gabriel spectrum.

Example 2: Let $R = \Lambda$ be an artin algebra. Such a ring is both left coherent and left semiartinian. If E is an indecomposable injective module, then the point $(-, E)$ is isolated in the contravariant Ziegler spectrum of Λ. By duality, E is a finitely presented module so the point $- \otimes_\Lambda E$ is isolated ([30, Corollary 13.4]) in the covariant Ziegler spectrum of Λ. It follows that the canonical bijective correspondence Ξ_Λ between the covariant and contravariant spectra of R is a homeomorphism with respect to the Ziegler topology as well as the Gabriel topology.

There is another homeomorphism between the Gabriel (resp., Ziegler) spectra of Λ, namely, that induced by the duality $D : (\Lambda\text{-mod})^{\mathrm{op}} \to \text{mod-}\Lambda$. In general, it is not the same as Ξ_Λ.

Example 3: Let $R = \mathbb{Z}$ be the ring of integers; it is a coherent ring. The indecomposable injective $\mathbb{Z}$-modules that are envelopes of simple modules are the Prüfer groups $\mathbb{Z}(p^\infty)$. It is well-known (cf. [30, Example 1, p. 104]) that the corresponding points $- \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty)$ are not isolated in the Ziegler spectrum of $\mathbb{Z}$. Therefore, the canonical bijective correspondence $\Xi_{\mathbb{Z}}$ is not a homeomorphism of Ziegler spectra.

Example 4: Let R be a **von Neumann regular** ring. This means that R satisfies any (all) of the following equivalent conditions:

(1) every short exact sequence of R-modules is pure-exact;
(2) every module is flat;
(3) every module is fp-injective;
(4) every pure-injective module is injective;
(5) every finitely presented module is projective.

The last condition of the proposition is equivalent to the equation $R\text{-}\underline{\text{mod}} = 0$, which the Auslander-Bridger transpose shows to be left-right symmetric. Thus $((R\text{-}\underline{\text{mod}})^{\mathrm{op}}, \mathrm{Ab}) = 0$ and the evaluation functor $\mathrm{Ev}\colon ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab}) \to R\text{-Mod}$ is an equivalence. The section morphism $\Upsilon : R\text{-Mod} \to ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ given by $M \mapsto (-, M)$ is its equivalence inverse. That the morphism Υ is an equivalence implies that for a von Neumann regular ring R the functorial perspective considered in this article coincides with the classical approach to the study of R-modules.

Example 5: Let R be a ring with the property that every submodule of a flat module is flat. Equivalently, the category R-Mod has flat global dimension at most 1. This is a left-right symmetric condition, because it is equivalent to $\mathrm{Tor}_2^R(M, N) = 0$ for any right R-module M and left R-module N.

PROPOSITION 50: *The category* R-Mod *has flat global dimension at most* 1 *if and only if the torsion class* $\mathrm{Gen}(-, R)$ *is hereditary.*

Proof. Suppose that R-Mod has flat global dimension at most 1. It suffices to prove that every subfunctor F of a coproduct $(-,R)^{(\alpha)} \cong (-,R^{(\alpha)})$ itself belongs to $\mathrm{Gen}(-,R)$. So let $\eta : (-,M) \to F$ be an epimorphism from a projective functor $(-,M)$. Composing the epimorphism with the inclusion $F \subseteq (-,R^{(\alpha)})$ yields a morphism of the form $(-,f) : (-,M) \to (-,R^{(\alpha)})$, where $f : M \to R^{(\alpha)}$ is morphism of R-modules. By assumption, the image of f is a flat module Z, so that $F = \mathrm{Im}\,(-,f) \cong (-,\mathrm{Im}\, f) = (-,Z)$. But $(-,Z)$ belongs to $\mathrm{Gen}(-,R)$, since any epimorphism $g : R^{(\beta)} \to Z$ from a free module to a flat module induces an epimorphism $(-,g) : (-,R)^{(\beta)} \to (-,Z)$.

To prove the converse, consider an inclusion $Y \subseteq Z$ of R-modules, with Z flat. Since $(-,Z)$ belongs to $\mathrm{Gen}(-,R)$, and $(-,Y) \subseteq (-,Z)$, the assumption that $\mathrm{Gen}(-,R)$ is hereditary implies that $(-,Y)$ also belongs to $\mathrm{Gen}(-,R)$. Thus there is an epimorphism $\eta : (-,R^{(\beta)}) \cong (-,R)^{(\beta)} \to (-,Y)$, which by Proposition 2 implies η is of the form $(-,g)$ where $g : R^{(\beta)} \to Y$ is a pure-epimorphism. But then Y is flat. ∎

Let us note that the open subset of the contravariant Gabriel spectrum $\mathrm{Sp}((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ that corresponds to the hereditary torsion class $\mathrm{Gen}(-,R)$ is the subset $\mathrm{Sp}(R\text{-Mod})$. It is clear that if E is an indecomposable injective R-module, then there is a nonzero morphism from $(-,R)$ to $(-,E)$. Thus $\mathrm{Sp}(R\text{-Mod})$ is contained in the open subset associated to $\mathrm{Gen}(-,R)$. To prove that equality holds, suppose that $I = E(\mathrm{Ext}^1(-,M))$, is a point of the contravariant Gabriel spectrum of R that does not belong to $\mathrm{Sp}(R\text{-Mod})$. If there were a nonzero morphism from $(-,R)$ to I, then, as $\mathrm{Gen}(-,R)$ is hereditary and $\mathrm{Ext}^1(-,M)$ is essential in I, it would lead to the contradiction that there exists a nonzero morphism from $(-,R)$ to $\mathrm{Ext}^1(-,M)$.

Example 6: Let R be a **left semi-hereditary** ring, that is, suppose that every finitely generated left ideal of R is projective. Then (cf. [33, Prop. I.6.9]), every finitely generated submodule of a finitely generated free module is projective. Equivalently, every finitely presented R-module has projective dimension at most 1. This is expressible by the equation in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$,

$$\mathrm{Ext}^2(-,M) = 0,$$

for every R-module M. Since every finitely presented module A has projective dimension at most 1, it also has flat dimension at most 1. For any right R-module W, we thus obtain that $\mathrm{Tor}_2^R(W,A) = 0$. As every left R-module is a direct limit of finitely presented modules, and the functor $\mathrm{Tor}^2_R(W,-) : R\text{-Mod} \to \mathrm{Ab}$ commutes with direct limits, it follows that for every left semi-hereditary ring R, the category R-Mod has flat global dimension at most 1. Indeed, it is not difficult to see that a ring R is left semi-hereditary if and only if it is left coherent and R-Mod has flat global dimension at most 1.

PROPOSITION 51: *If R is a left semi-hereditary ring, then the subset* $\mathrm{Sp}(R\text{-Mod})$ *of the covariant Gabriel spectrum* $\mathrm{Sp}(\text{mod-}R, \mathrm{Ab})$ *is open.*

Proof. First, note that any quotient of an injective module E is fp-injective. For if $M \subseteq E$ is a submodule, then a routine dimension shift argument implies that $\mathrm{Ext}^1(-, E/M) \cong \mathrm{Ext}^2(-, M) = 0$ in $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$.

Now suppose that there is an indecomposable injective module E such that the point $- \otimes E$ of the covariant Gabriel spectrum belongs to the closure of $\mathrm{Sp}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$. There is an embedding $- \otimes m : - \otimes E \to \prod_i - \otimes U_i$, where each $- \otimes U_i$ belongs to $\mathrm{Sp}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$. The R-morphism $m : E \to \prod_i U_i$ is an embedding, so that one of the component functions $m_i : E \to U_i$ is nonzero. The image of m_i is fp-injective, which implies that U_i must be the pure-injective envelope of an fp-injective module. But then U_i would have to be injective, contradicting that $- \otimes U_i$ belongs to $\mathrm{Sp}(\underline{\mathrm{mod}}\text{-}R, \mathrm{Ab})$. ∎

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